Operator algebraic quantum groups

The problem is the antipode

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Outline

- Introduction
- Duality for Hopf * algebras
- Generalization to locally compact groups
- The counit and the antipode. Problems
- The antipode. Solution
- Examples
- Conclusion and some references

Introduction

Operator algebraists became interested in the theory of quantum groups long before the notion became popular with the work of Drinfel'd in 1986.

The motivation grew out of the attempts to generalize Pontryagin's duality theorem from locally compact abelian groups to the non abelian case. Recall the result.

Theorem (Pontryagin)

Let G be a locally compact abelian group. The set of characters $\widehat{G}$ has a natural topology making it again into a locally compact abelian group. It is called the dual of G. The dual of $\widehat{G}$ is naturally isomorphic with the original group G.

The result dates from as early as 1939. It is the key result for Abstract Harmonic Analysis.

Introduction - cont'd

To generalize this result, the idea is simple. Given say a finite group G, we associate a dual pair of Hopf algebras. One is the group algebra $\mathbb{C}G$ and the other one, its dual, the function algebra $F(G)$.

For an abelian group, this duality is essentially the same a Pontryagin's duality.

Therefore, in order to generalize Pontryagin's duality, we must look for a more general concept than a Hopf algebra. We have to find the analogues of the Hopf algebras $F(G)$ and $\mathbb{C}G$ for locally compact, possibly non-abelian groups and a duality between the two.

In fact, this is not so difficult to find in the framework of operator algebras. The main difficulty comes from the desire to get objects containing the two cases, together with a duality.

This leads to some natural constraints.

Introduction - cont'd

Working with operator algebras means that we only look a algebras over $\mathbb{C}$ with an involution, satisfying

$$a^*a = 0 \quad \text{only if} \quad a = 0.$$

We will need integrals in order to be able to construct the dual.

And of course, it will no longer be expected that the coproduct maps into an algebraic tensor product of the algebra.

All these constraints turn out to be rather easy to overcome using notions and results, familiar in circles of functional analysts. However, ... except for the antipode! There is now a satisfactory solution, but it took quite some time before this was achieved.

I will try to explain why this is a problem and how it has been solved.

Duality for Hopf *-algebras

The following result is well-known.

Proposition

Let (A, Δ) be a finite-dimensional Hopf *-algebra. Then the dual space A' is again a Hopf *-algebra with the product and coproduct defined by

$$(\omega\omega')(a) = (\omega \otimes \omega')\Delta(a) \qquad \Delta(\omega)(a \otimes a') = \omega(aa').$$

The involution on the dual is given by $\omega^* = \overline{\omega(S(a)^*)}$ where S is the antipode of (A, Δ).

The unit in A' is given by the counit of A and the counit on A' is obtained by evaluation in the identity of A. Finally, the antipode S on A' is given by $(S(\omega))(a) = \omega(S(a))$.

The dual of A' is again A.

Dual pairs of Hopf *-algebras

Definition (A dual pair of Hopf *-algebras)

Let (A, Δ) and (B, Δ) be Hopf *-algebras. Let $(a, b) \mapsto \langle a, b \rangle$ be a non-degenerate bilinear form $A \times B$ to $\mathbb{C}$. Assume that

$$\langle \Delta(a), b \otimes b' \rangle = \langle a, bb' \rangle \qquad \langle a \otimes a', \Delta(b) \rangle = \langle aa', b \rangle$$

and that $\langle a, b^* \rangle = \overline{\langle S(a)^*, b \rangle}$.

Proposition

Let G be a finite group. Let A be the function algebra and B the group algebra. Denote by $p \mapsto \lambda_p$ the embedding of G in B. Then $\langle f, \lambda_p \rangle = f(p)$ makes (A, B) into a dual pair of Hopf *algebras.

The result can be generalized to infinite groups using multiplier Hopf *-algebras.

Duality for locally compact groups

Consider a locally compact group G. We associate two *-algebras. For A we take the algebra $C_c(G)$ of continuous functions with compact support and pointwise operations. The involution $f \mapsto f^*$ is given by $f^*(p) = \overline{f(p)}$.

For B we again take continuous functions with compact support but now with convolution product

$$(g_1 g_2)(p) = \int g_1(q) g_2(q^{-1}p)\, dq.$$

We integrate over the left Haar measure. For the involution here we take

$$g^*(p) = \delta_G(p)^{-1}\overline{g(p^{-1})}.$$

Here δ_G is the modular function of G. The pairing is given by

$$\langle f, g \rangle = \int f(p) g(p)\, dp.$$

The coproduct on the function algebra $C_c(G)$

The coproduct on $C_c(G)$ induced by the pairing is a *-homomorphism from $C_c(G)$ to the algebra $C(G \times G)$ of continuous functions on $G \times G$ given by

$$\Delta(f)(p,q) = f(pq).$$

Remark that this is no longer a map with values in the algebraic tensor product of $C_c(G)$ with itself. There is a counit given by $\varepsilon(f) = f(e)$. This is a *-homomorphism.

There is an antipode S given by $S(f)(p) = f(p^{-1})$. We still can say that

$$m(S \otimes \iota)\Delta(f) = \varepsilon(f)1 \qquad m(\iota \otimes S)\Delta(f) = \varepsilon(f)1$$

(where ι is the identity map and m is multiplication) but we need to explain these formulas.

The formula $m(S \otimes \iota)\Delta(f) = \varepsilon(f)1$

Fix $f \in C_c(G)$. Then $\Delta(f)$ is a continuous function on $G \times G$. We can apply $S \otimes \iota$ to such functions. We have

$$((S \otimes \iota)F)(p, q) = F(p^{-1}, q).$$

It is again a function in $C(G \times G)$. Now we want to apply multiplication. We have

$$(m(f_1 \otimes f_2))(p) = (f_1 f_2)(p) = f_1(p) f_2(p) = (f_1 \otimes f_2)(p, p).$$

So we define m on elements in $C(G \times G)$ by $(m(F))(p) = F(p, p)$. Combining all this we find

$$\begin{aligned}(m(S \otimes \iota)\Delta(f))(p) &= ((S \otimes \iota)\Delta(f))(p, p) \\ &= \Delta(f)(p^{-1}, p) \\ &= f(p^{-1}p) = f(e) = \varepsilon(f).\end{aligned}$$

Hence we get $m(S \otimes \iota)\Delta(f) = \varepsilon(f)1$.

Problems

Suppose we want to generalize this. We take a C*-algebra A instead of $C_c(G)$. The notion of a coproduct on a C*-algebra is well understood. It is a *-homomorphism from A to the multiplier algebra $M(A\bar{\otimes}A)$ of a completed tensor product $A\bar{\otimes}A$.

The counit is already problematic because it can not be assumed to be continuous.

However, the main problem is the antipode.

- It can not be assumed to be a continuous anti-isomorphism.
- We can not properly define the maps $S \otimes \iota$ and $\iota \otimes S$, not even on the algebraic tensor product $A \otimes A$ and certainly not on the completion $A\bar{\otimes}A$.
- Also the multiplication map is not continuous in general.

So a formula like $m(S \otimes \iota)\Delta(a) = \varepsilon(a)1$ simply makes no sense!

Solution

For the solution we use the following result for finite Hopf algebras (A, Δ).

Proposition

For any $a \in A$ we can write (in a unique way)

$$a \otimes 1 = \sum_i \Delta(p_i)(1 \otimes q_i)$$

and then

$$S(a) \otimes 1 = \sum_i (1 \otimes p_i)\Delta(q_i).$$

In a Hopf *-algebra we use

$$a \otimes 1 = \sum \Delta(p_i)(1 \otimes q_i^*) \qquad S(a)^* \otimes 1 = \sum \Delta(q_i)(1 \otimes p_i^*).$$

In the case of a locally compact group

Take a locally compact group G and $f \in C_c(G)$. Write $f(p) = f(pqq^{-1})$ and approximate

$$f((pq)q^{-1}) \simeq \sum f_i(pq)g_i(q^{-1}) = \sum (\Delta(f_i))(p,q)(S(g_i))(q).$$

Then we get $\sum f_i(q)\Delta(S(g_i))(p,q) =$

$$\sum f_i(q)g_i(q^{-1}p^{-1}) \simeq f(qq^{-1}p^{-1}) = f(p^{-1}).$$

This is the idea to get the antipode in the general case of a coproduct Δ on C*-algebra A.

- Look for elements $a \in A$ so that there is an element $b \in A$ satisfying

$$a \otimes 1 \simeq \sum \Delta(p_i)(1 \otimes q_i) \qquad b \otimes 1 \simeq \sum (1 \otimes p_i)\Delta(q_i).$$

- Show that b is unique if it exists, that there are enough such elements a, and define $S(a) = b$.

Why all this trouble? An example

Consider the $ax + b$-group with matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \, a \neq 0, b \in \mathbb{R} \right\}$$

We can consider polynomial functions and deform this to a Hopf algebra:

Proposition

Take any non-zero complex number λ. Let A be the unital algebra generated by an invertible element a and an element b satisfying $ab = \lambda ba$. It is a Hopf algebra if we let

$$\Delta(a) = a \otimes a \qquad \text{and} \qquad \Delta(b) = a \otimes b + b \otimes 1.$$

Why is this not good enough?

No integrals and hence no duality!

Two subgroups and bicrossproducts

Take a look at the two subgroups:

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| \, a \neq 0, \right\}$$

$$K = \left\{ \begin{pmatrix} 1+b & b \\ 0 & 1 \end{pmatrix} \middle| \, b \neq -1 \right\}.$$

Clearly $H \cap K$ is trivial. We have almost a matched pair. We have e.g.

$$HK = \left\{ \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix} \middle| \, p \neq 0, q \neq p \right\}$$

The set HK is a proper but dense open subset of G. The map $(h, k) \mapsto hk$ is a homeomorphism of $H \times K$ to HK. One can associate a dual pair of locally compact quantum groups with special properties.

Conclusion

- The interest of functional analysts comes from the attempts to generalize Pontryagin's duality to non-abelian groups.
- This requires a topological version of a Hopf algebra, in the framework of operator algebras.
- The original characterization of the antipode with the formulas like $m(S \otimes \iota)\Delta(a) = \varepsilon(a)1$ faces fundamental problems.
- Another characterization using the canonical maps, and based on the Larsen-Sweedler theorem (providing the antipode from the integrals) solves these problems.
- This all results in a rich theory, with nice examples and phenomena that are not encountered in the purely algebraic theory of duality for Hopf algebras.

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