Stabilised Profunctors between Groupoids

Marcelo Fiore
University of Cambridge

SECOND ANTIPODE WORKSHOP
12. IX. 2022

Abstract

We examine the fundamental mathematical setting provided by the biequivalence between the compact closed bicategory Prof of profunctors (aka bimodules or distributors) between groupoids and natural transformations between them, and the 2-category of cocontinuous functors between categories of presheaves over groupoids (aka categories of groupoid actions) and natural transformations between them.

Our starting point is a basic representation theorem for presheaves over groupoids that leads to the consideration of groupoids with additional structure called kits. Kits have both combinatorial and logical character. From a combinatorial viewpoint, they serve to restrict presheaves to stable ones. From a logical perspective, we will consider a class of Boolean kits. These are drawn from Boolean algebras associated to groupoids by means of a general universal construction to be introduced and discussed. In this context, the dualities of profunctors and of Boolean algebras will be placed side by side to define a bicategory SProf of stabilised profunctors between Boolean kits and natural transformations between them. We shall show that SProf is $\star$-autonomous, with a projection onto Prof degenerating to its compact-closed structure, and that it is biequivalent to the 2-category of linear functors (those being left and right local adjoints) between categories of stable presheaves over Boolean kits and natural transformations between them.

The motivation and context for these investigations are developments in category theory (polynomial functors), structural combinatorics (species of structures), and theoreticalcomputer science (linear logic). In particular, symmetric-algebra structure (Fock space) in this setting plays a fundamental role and, time permitting, will also be discussed.

This is joint work with Zeinab Galal and Hugo Paquet.

Background

- Groupoids

  = small categories with all maps invertible

(every groupoid is equivalent to a small sum of groups)

Background

- Groupoids
  = small categories with all maps invertible
  (every groupoid is equivalent to a small sum of groups)

- Profunctors (bimodules or distributors)
  $P : \mathbb{A} \to\!\!\!\!\!+ \mathbb{B}$
  $= P : \mathbb{B}^{op} \times \mathbb{A} \to \underline{\underline{Set}}$ (example: $\hom_{\mathbb{A}} : \mathbb{A} \to\!\!\!\!\!+ \mathbb{A}$)
  $= P : \mathbb{A} \to \underline{\underline{Set}}^{\mathbb{B}^{op}}$ (example: $y_{\mathbb{A}} : \mathbb{A} \hookrightarrow Set^{\mathbb{A}^{op}}$)

- Presheaves (or actions)

$$\underline{\mathrm{Psh}}(\mathbb{A}) = \underline{\mathrm{Set}}^{\mathbb{A}^{op}}$$

examples: for $a \in \mathbb{A}$

- free actions: $y(a) \in \underline{\underline{\mathrm{Psh}}}(\mathbb{A})$

- **Presheaves (or actions)**

$\underline{\underline{Psh}}(\mathbb{A}) = \underline{\underline{Set}}^{\mathbb{A}^{op}}$

examples: for $a \in \mathbb{A}$

- free actions: $y(a) \in \underline{\underline{Psh}}(\mathbb{A})$
- quotients of free actions by subgroups:

  for $G \leq \mathbb{A}(a,a)$

  $$(g \in G) \circlearrowleft y(a) \twoheadrightarrow y(a)/G$$

- sums

Prof between groupoids

$$\underset{\text{profunctor}}{A \nrightarrow B} \simeq \underline{\underline{\mathrm{PSh}}}(A) \xrightarrow{\text{cocontinuous}} \underline{\underline{\mathrm{PSh}}}(B)$$

$\wr$

($\underline{\underline{\mathrm{PSh}}}(\mathbb{C})$ is the cocompletion of $\mathbb{C}$)

$\underline{\underline{\text{Prof}}}$ between groupoids

- $\mathbb{A} \overset{\text{profunctor}}{\nrightarrow} \mathbb{B} \quad \simeq \quad \underline{\underline{Psh}}(\mathbb{A}) \xrightarrow{\text{cocontinuous}} \underline{\underline{Psh}}(\mathbb{B})$

  $\wr$

  ($\underline{\underline{Psh}}(\mathbb{C})$ is the cocompletion of $\mathbb{C}$)

- $\hom_{\mathbb{A}} \simeq \mathrm{Id}_{\underline{\underline{Psh}}(\mathbb{A})}$

- composition

$$\frac{\mathbb{A} \overset{P}{\nrightarrow} \mathbb{B} \overset{Q}{\nrightarrow} \mathbb{C}}{\mathbb{A} \underset{Q\circ P}{\nrightarrow} \mathbb{C}}$$

$$(Q\circ P)(c,a) = \left(\sum_{b\in\mathbb{B}} Q(c,b)\times P(b,a) \right) \Big/ \simeq$$

Prof

- symmetric monoidal closed structure

$$\mathbb{A} \otimes \mathbb{B} = \mathbb{A} \times \mathbb{B}$$

- biproduct structure

$$\mathbb{A} \oplus \mathbb{B} = \mathbb{A} + \mathbb{B}$$

- compact closed structure

$$\mathbb{A}^* = \mathbb{A}^{op}$$

- symmetric-algebra structure (or Fock space)

$S(\mathbb{A})$ = symmetric-monoidal completion

Representation of groupoid actions

- Every $X \in \underline{\mathrm{PSh}}(\mathbb{A})$ is of the form

$$\sum_{i \in I} y(a_i)/G_i$$

where

$$G_i \in \{ \underline{\mathrm{Stab}}(x) \mid x \in X(a_i) \}$$

$$\underline{\mathrm{Stab}}(x) = \{ \alpha \in \mathbb{A}(a_i, a_i) \mid x \cdot \alpha = x \}$$

Representation of groupoid actions

- Every $X \in \underline{\mathrm{PSh}}(\mathbb{A})$ is of the form

$$\sum_{i \in I} y(a_i)/G_i$$

where

$$G_i \in \{ \underline{\mathrm{Stab}}(x) \mid x \in X(a_i) \}$$

$$\underline{\mathrm{Stab}}(x) = \{ \alpha \in \mathbb{A}(a_i, a_i) \mid x \cdot \alpha = x \}$$

$\underline{\underline{\text{NB}}}$: $\forall a \in \mathbb{A}.\ \{ \underline{\mathrm{Stab}}(x) \mid x \in X(a) \}$ is closed under conjugation

Kits on groupoids

- A **kit** $\mathcal{A}$ on $\mathbb{A}$ is an assignment

$$a \in \mathbb{A} \longmapsto \mathcal{A}(a) \subseteq \underline{\text{SubGroups}}\big(\mathbb{A}(a,a)\big)$$

closed under conjugation

Example: for $X \in \underline{\mathrm{Psh}}(\mathbb{A})$

$$a \longmapsto \{\, \underline{\mathrm{Stab}}(x) \mid x \in X(a) \,\}$$

Stable presheaves

$$\mathcal{S}(\mathbb{A}, \mathscr{A}) \hookrightarrow \underline{\mathrm{PSh}}(\mathbb{A})$$

$$\|$$

$$\{ X \mid \forall a \in \mathbb{A}.\ \forall x \in X(a).\ \underline{\mathrm{Stab}}(x) \in \mathscr{A}(a) \}$$

Boolean kits via orthogonality

- Let A be a group. For $G, G' \leq A$ define

$$G \perp G' \iff G \cap G' \text{ is trivial}$$

Boolean kits via orthogonality

- Let A be a group. For $G, G' \leq A$ define

$$G \perp G' \iff G \cap G' \text{ is trivial}$$

- The orthogonal of a kit $\mathscr{A}$ on $\mathbb{A}$ is the kit $\mathscr{A}^\perp$ on $\mathbb{A}^{op}$ defined as:

$$\mathscr{A}^\perp(a) = (\mathscr{A}(a))^\perp$$

$$= \{ G' \leq \mathbb{A}(a,a) \mid \forall G \in \mathscr{A}(a).\ G \perp G' \}$$

Boolean Kits

- Boolean kits are those such that

$$\mathcal{A} = \mathcal{A}^{\perp\perp}$$

imposes strong closure properties: non-emptiness, downwards closure, closure under filtered unions

Boolean Kits

- Boolean kits are those such that

$$\mathcal{A} = \mathcal{A}^{\perp\perp}$$

imposes strong closure properties: non-emptiness, downwards closure, closure under filtered unions

- The Boolean kits on a groupoid form a complete Boolean algebra with complement given by orthogonality and meets by intersection

(Rem: This construction arises by Booleanisation)

Stabilised profunctors between Boolean kits

- $P : (A, \mathcal{A}) \nrightarrow (B, \mathcal{B})$

$=$ $P : A \nrightarrow B$

s.t.

$$\alpha \cdot p \cdot \beta = p \in P(b, a)$$

implies

$$\alpha \in \bigcup \mathcal{A}(a) \Rightarrow \beta \in \bigcup \mathcal{B}(b)$$

and

$$\beta \in \bigcup \mathcal{B}^\perp(b) \Rightarrow \alpha \in \bigcup \mathcal{A}^\perp(a)$$

Stabilised profunctors between Boolean kits

- $P : (A, \mathcal{A}) \nrightarrow (B, \mathcal{B})$

$=$ $P : A \nrightarrow B$

s.t.

$$\alpha \cdot p \cdot \beta = p \in P(b, a)$$

implies

$$\alpha \in \bigcup \mathcal{A}(a) \Rightarrow \beta \in \bigcup \mathcal{B}(b)$$

and

$$\beta \in \bigcup \mathcal{B}^{\perp}(b) \Rightarrow \alpha \in \bigcup \mathcal{A}^{\perp}(a)$$

<u>Rem</u>: Stabilised profunctors are closed under composition

Rem:

$$P : (A, \mathcal{A}) \nrightarrow (B, \mathcal{B})$$

ff

$$\begin{tikzcd}
S(A, \mathcal{A}) \arrow[r, dashed] \arrow[d, hook] & S(B, \mathcal{B}) \arrow[d, hook] \\
\underline{\mathrm{Psh}}(A) \arrow[r, "P^{\#}"'] & \underline{\mathrm{Psh}}(B)
\end{tikzcd}$$

and

$$\begin{tikzcd}
S(B^{\circ}, \mathcal{B}^{\perp}) \arrow[r, dashed] \arrow[d, hook] & S(A^{\circ}, \mathcal{A}^{\perp}) \arrow[d, hook] \\
\underline{\mathrm{Psh}}(B^{\circ}) \arrow[r, "(P^{*})^{\#}"'] & \underline{\mathrm{Psh}}(A^{\circ})
\end{tikzcd}$$

$$\dfrac{(A, \mathcal{A}) \xrightarrow[t]{\text{stabilised profunctor}} (B, \mathcal{B})}{\overline{\overline{S(A, \mathcal{A}) \xrightarrow[\text{linear functor}]{} S(B, \mathcal{B})}}}$$

//

left and right local adjoint

SProf

- symmetric monoidal closed structure

$$(A, \mathcal{A}) \otimes (B, \mathcal{B}) = (A \times B, \ (\mathcal{A} \times \mathcal{B})^{\perp\perp})$$

- biproduct structure

$$(A, \mathcal{A}) \oplus (B, \mathcal{B}) = (A + B, \ [\mathcal{A}, \mathcal{B}])$$

- $*$-autonomous structure

$$(A, \mathcal{A})^* = (A^{op}, \mathcal{A}^{\perp})$$

- symmetric-algebra structure (or Fock space)

$$\Sigma(A, \mathcal{A}) = (S(A), \ \sigma(\mathcal{A})^{\perp\perp})$$

Generalised species of structures

▶ $S(\mathbb{A}) \nrightarrow \mathbb{B}$ generalised species $\underline{\mathbf{Esp}}$

Generalised species of structures
and
analytic functors

$$\frac{S(A) \longrightarrow B}{\underline{\mathrm{Psh}}(A) \longrightarrow \underline{\mathrm{Psh}}(B)}$$

$$\frac{\text{generalised species} \quad \underline{\underline{\mathrm{Esp}}}}{\text{analytic functors} \quad \underline{\underline{\mathrm{Ana}}}}$$

(= finitary preserving quasi-pullbacks and inverse limits)

Generalised species of structures
and
analytic functors

- $$\frac{S(\mathbb{A}) \mathrel{\mapstochar\longrightarrow} \mathbb{B}}{\underline{\mathrm{PSh}}(\mathbb{A}) \longrightarrow \underline{\mathrm{PSh}}(\mathbb{B})}
  \qquad
  \frac{\text{generalised species} \quad \underline{\underline{\mathrm{Esp}}}}{\text{analytic functors} \quad \underline{\underline{\mathrm{Ana}}}}$$

- $\underline{\underline{\mathrm{Esp}}}/\underline{\underline{\mathrm{Ana}}}$ have cartesian closed and differential structure

Stable species and functors

- $$\begin{array}{c} \Sigma(\mathbb{A},\mathcal{A}) \xrightarrow{t} (\mathbb{B},\mathcal{B}) \\ \hline\hline \mathcal{S}(\mathbb{A},\mathcal{A}) \longrightarrow \mathcal{S}(\mathbb{B},\mathcal{B}) \end{array}$$

  $$\begin{array}{c} \text{stable species} \\ \hline\hline \text{stable functors} \end{array} \qquad \begin{array}{c} \underline{\underline{\mathbf{SEsp}}} \\ \\ \underline{\underline{\mathbf{Stab}}} \end{array}$$

  (= finitary, epi-preserving local right adjoints)

- $\underline{\underline{\mathbf{SEsp}}}$ / $\underline{\underline{\mathbf{Stab}}}$ have cartesian closed and differential structure

References

Groupoids

P J Higgins. Categories and Groupoids. Reprints in Theory and Applications of Categories, No. 7, pp 1–195, 2005.

Profunctors

N Yoneda. On Ext and exact sequences. Journal of the Faculty of Science, Imperial University of Tokyo, 8:507–576, 1960.

J Bénabou. Distributors at work. Lecture notes written by Thomas Streicher. URL: https://www2.mathematik.tu-darmstadt.de/~streicher/FIBR/DiWo.pdf, 2000.

F W Lawvere. Metric spaces, generalized logic, and closed categories. Rendiconti del Seminario Matematico e Fisico di Milano, 43:135–166, 1973. Republished in: Reprints in Theory and Applications of Categories, No. 1, pp 1–37, 2002.

B Day and R Street. Monoidal bicategories and Hopf algebroids. Advances in Mathematics, 129(1):99–157, 1997.

Presheaves

S Mac Lane and I Moerdijk. Sheaves in Geometry and Logic: A First Introduction to Topos Theory. Springer Universitext, 1994.

Combinatorial Fock Space

M Fiore. An Axiomatics and a Combinatorial Model of Creation/Annihilation Operators. In arXiv <arxiv.org/abs/1506.06402>, 2015.

Species of Structures and Analytic Functors

A Joyal. Foncteurs analytiques et espèces de structures. In Combinatoire énumérative, Springer LNM 1234, pages 126–159, 1986.

M Fiore. Mathematical models of computational and combinatorial structures. In International Conference on Foundations of Software Science and Computation Structures, pages 25–46. Springer, 2005.

M Fiore, N Gambino, M Hyland, and G Winskel. The cartesian closed bicategory of generalised species of structures. Journal of the London Mathematical Society (2), 77(1):203–220, 2008.

M Fiore. Analytic functors between presheaf categories over groupoids. Theoretical Computer Science, 546:120–131, 2014.

N Gambino and A Joyal. On operads, bimodules and analytic functors. Memoirs of the American Mathematical Society, Volume 249, Number 1184, 2017.

M Fiore, N Gambino, M Hyland, and G Winskel. Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures. Selecta Mathematica, 24(3):2791–2830, 2018.

Kits

M Fiore. Stable species of structures. Workshop on Polynomial Functors (https://www.youtube.com/watch?v=RDuNIP4icKI&t=70s), 2021.

M Fiore, Z Galal, and H Paquet. A combinatorial approach to higher-order structure for polynomial functors. In 7th International Conference on Formal Structures for Computation and Deduction (FSCD), 2022.