Stabilised Profunctors between Groupoids

Morcelo Fiore University of Combridge

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Abstract

We examine the fundamental mathematical setting provided by the biequivalence between the compact closed bicategory Prof of profunctors (aka bimodules or distributors) between groupoids and natural transformations between them, and the 2-category of cocontinuous functors between categories of presheaves over groupoids (aka categories of groupoid actions) and natural transformations between them.

Our starting point is a basic representation theorem for presheaves over groupoids that leads to the consideration of groupoids with additional structure called kits. Kits have both combinatorial and logical character. From a combinatorial viewpoint, they serve to restrict presheaves to stable ones. From a logical perspective, we will consider a class of Boolean kits. These are drawn from Boolean algebras associated to groupoids by means of a general universal construction to be introduced and discussed. In this context, the dualities of profunctors and of Boolean algebras will be placed side by side to define a bicategory SProf of stabilised profunctors between Boolean kits and natural transformations between them. We shall show that SProf is *-autonomous, with a projection onto Prof degenerating to its compact-closed structure, and that it is biequivalent to the 2-category of linear functors (those being left and right local adjoints) between categories of stable presheaves over Boolean kits and natural transformations between them.

The motivation and context for these investigations are developments in category theory (polynomial functors), structural combinatorics (species of structures), and theoreticalcomputer science (linear logic). In particular, symmetric-algebra structure (Fock space) in this setting plays a fundamental role and, time permitting, will also be discussed.

This is joint work with Zeinab Galal and Hugo Paquet.

Beckground Groupvids = small categories with all maps inverbible (every groupoid is equivalent to a small sum of groups)

► Presheaves (or actions)

Psh(A) = Set APP examples: for aEA • free actions: y(a) E PSh (A)

► Presheaves (or actions) Psh (A) = Set APP examples: for aEA • free actions: y(a) E PSh (A) · quotients of free actions by subgroups: for G<A(a,a) (geg) Gyan >> yan/G



Prof between groupoids profunctor cocontinuous $A \rightarrow B \approx P8h(A) \longrightarrow P8h(B)$ 7 (PSh(C) is The completion of C)

Prof between groupoids Profunctor cocontinuous $A + B \approx P8h(A) \longrightarrow P8h(B)$ (PSh(c) is The cromple birt of C) homA ~ Idpsh(A) $A \rightarrow B \rightarrow C$ · composition QoP $(Q_0 \hat{P})(c, a) = \left(\sum_{b \in B} Q(c, b) \times P(b, a)\right) \approx$

Prof

> symme tric monoridal closed structure $A \otimes B = A \times B$ ► biproduct structure $A \oplus B = A + B$ · compact closed structure A* = A sp ► symmetriz-algebra structure (or Fock space) S(A) = symmetriz-monoridal completion

Representation of groupsid actions Every XE PSh(A) is of the form $\sum_{i \in I} y(ai)/Gi$ where Gie $\begin{cases} Stab(z) \mid z \in X(ai) \end{cases}$ \parallel $\int \alpha \in \mathcal{A}(a_i, a_i) \mid \mathbf{x} \cdot \mathbf{x} = \mathbf{x}$

Representation of groupsid actions Every XE PSh(A) is of the form $\sum_{i \in I} \frac{y(ai)}{6i}$ where Gie $\{ Stab(z) \mid z \in X(ai) \}$ $\|$ $\int d \in A(a_i, a_i) | \mathbf{x} \cdot \mathbf{x} = \mathbf{x}$ $NB: \forall a \in A. \{ Stab(z) \mid z \in X(a) \}$ is closed under conjugation

Kits on groupoids

A kit of on A is in assignment

Example: for
$$X \in P8h(A)$$

 $a \mapsto f Stob(a) | x \in X(a)$

Stable presheaves $S(A, A) \hookrightarrow Pgh(A)$ $\{X \mid \forall a \in A. \forall x \in X(a). Stab(z) \in \mathcal{A}(a)\}$

Boolean Kits via orthogonality
Let A be a group. For G, G'≤A define
G⊥G' (=> G∩G' is trivial

Booleon kits via orthogonality ► Let A be a group. For G, G'SA define GIG' (=> GNG' is trivial The orthogonal of a kit of on 1A is the kit of on Asp defined 28: $aA^{+}(a) = (aA(a))^{\perp}$ = $\{G' \leq A(a,a)\} \forall G \in \mathcal{A}(a), G \perp G'\}$

Boolean Kits Boolean Kits are those such That $A = A^{\perp \perp}$ imposes strong closure properties: non-en pliness, down words closure, closure under filtered unions

Booleon Rits

 $d = d \perp 1$

Boolean kits are those such That

imposes strong closure properties : non-en pliness, downwards closure, closure under filtered unions

The Boolean kits on a groupoid form a complete Boolean algebra with complement given by orthogonaloty and meets by intersection (Rem: This construction arises by Booleanisation)

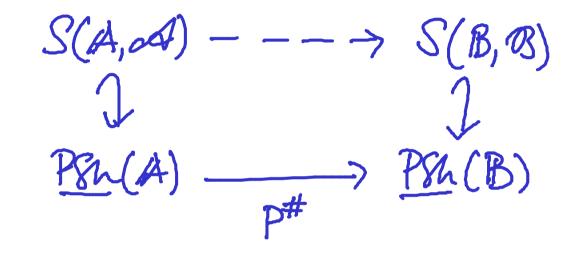
Stabilised profunctors between Boolean kits $\blacktriangleright P: (A, A) \leftrightarrow (B, B)$ $= P: A \rightarrow B$ st. $d \cdot p \cdot \beta = p \in P(b, a)$ implies $\alpha \in U \circ \mathcal{A}(\circ) \Rightarrow \beta \in U \mathcal{B}(5)$ and $\beta \in \mathcal{O} \mathcal{B}^{\perp}(b) \Rightarrow \alpha \in \mathcal{O} \mathcal{A}^{\perp}(a)$

Stabilised profunctors between Boolean kits $\blacktriangleright P: (A, A) \leftrightarrow (B, B)$ $= P: A \rightarrow B$ st. $d \cdot p \cdot \beta = p \in P(b, a)$ implies $d \in U \otimes (a) \Rightarrow \beta \in U \otimes (b)$ and $\beta \in \mathcal{OB}^{\perp}(b) \Rightarrow \alpha \in \mathcal{OAL}(a)$ Rem: Stabilised profunctors are closed under composition

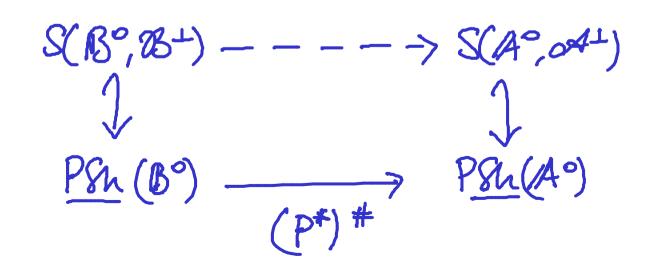
Kem:

 $P: (A, A) \rightarrow (B, B)$





and



stabilised (A, A) - t 7 (B, B)

S(A, A) _____ S(B, B) lineor functor // left and right local adjoint



Symmetric monoridal closed structure $(A, A) \otimes (B, B) = (A \times B, (A \times B)^{++})$ > biproduct structure $(A, A) \oplus (B, B) = (A + B, [A, B])$ ► * autonomous structure $(A, A)^{*} = (A^{\circ p}, A^{\perp})$ ► symmetriz-algebra structure (or Fock space) $\Sigma(A, aA) = (S(A), S(aA)^{\perp +})$

Generalised species of structures

 $S(A) \rightarrow B$

generalised species Esp

Generalised species of structures onalytic functors

 $S(A) \rightarrow B$ $Psh(A) \longrightarrow Psh(B)$

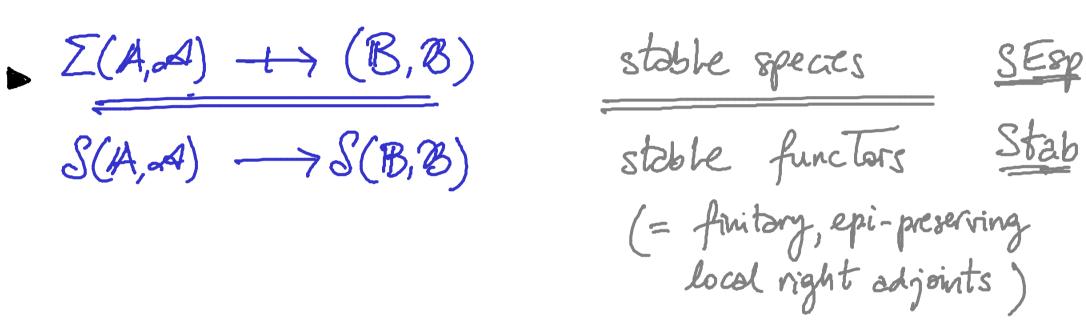
generalised species Esp malytic functors Ana (= finitary preserving quasi-pullbacks and inverse limits)

Generalised species of structures and onalytic functors

generalised species Esp $S(A) \rightarrow B$ onalytic functors Ana $Psh(A) \longrightarrow Psh(B)$

 Esp/And have cartesian closed and differential structure

Stable species and functors



Step/Step have cartesian closed and differential structure

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