Partial actions in semigroup theory

Ganna Kudryavtseva

University of Ljubljana
Faculty of Civil and Geodetic Engineering
IMFM, Ljubljana

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Partial actions of groups

Partial group actions: equivalent definitions

- G – group with the unit e, X – set
- Definition 1. A partial action of G on X is $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ where $X_g \subseteq X$ and $\alpha_g\colon X_{g^{-1}} \to X_g$ is a bijection $\forall g \in G$, such that:
  (i) $X_e = X$ and $\alpha_e = \mathrm{id}_X$
  (ii) $\alpha_h^{-1}(X_{g^{-1}} \cap X_h) \subseteq X_{(gh)^{-1}}$
  (iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for each $x \in \alpha_h^{-1}(X_{g^{-1}} \cap X_h)$
- A partial map $\varphi\colon A \to B$ is a map $C \to B$ where $C \subseteq A$. We say that $\varphi(a)$ is defined if $a \in C$ and undefined otherwise.
- Definition 2. A partial action of G on X is a partial map $*\colon G \times X \to X,\ \ (g, x) \mapsto g * x$ (whenever defined) such that
  (i) $e * x$ is defined and equals x for all $x \in X$.
  (ii) if $g * x$ is defined then $g^{-1} * (g * x)$ is defined and $g^{-1} * (g * x) = x$.
  (iii) if $h * x$ and $g * (h * x)$ are defined then $gh * x$ is defined and $g * (h * x) = gh * x$.

Partial group actions: equivalent definitions

- S – inverse monoid, e.g., $S = \mathcal{I}(X)$
- A premorphism $\varphi\colon G \to S$, $g \mapsto \varphi_g$, is a map such that

  (i) $\varphi_e = e$

  (ii) $\varphi_{g^{-1}} = (\varphi_g)^{-1}$

  (iii) $\varphi_g \varphi_h \leq \varphi_{gh}$
- Definition 3. A partial action of G on X is a premorphism $G \to \mathcal{I}(X)$.
- Definitions 1,2 and 3 are equivalent.
- Definition 1 was introduced by Exel in 1998. Definition 2 first appears in Kellendonk and Lawson (2004). Definition 3 first appears in the work by McAlister and Reilly in 1977, and then applied by Petrich and Reilly to the description of E-unitary inverse semigroups.
- Partial actions of groups are precisely restrictions of actions (Kellendonk, Lawson, 2004).

E-unitary inverse semigroups
via
partial actions of groups

Groupoid of a partial action of a group

- $*$ – partial action of G on X
- Objects of $\mathcal{G} = \mathcal{G}(G, X, *)$: elements of X
- There is an arrow $\underset{x}{\bullet} \xrightarrow{g} \underset{y}{\bullet}$ iff $g * x$ is defined and $g * x = y$. Denote this arrow by (y, g).
- $(z, h) \cdot (y, g)$ exists in $\mathcal{G}$ iff $h^{-1} * z = y$.

$$\underset{x}{\bullet} \xrightarrow{g} \underset{y}{\bullet} \xrightarrow{h} \underset{z}{\bullet}$$

  then $(z, h) \cdot (y, g) = (z, hg)$.
- Suppose that X is a semilattice and G acts partially on it by order izomorphisms between order ideals.
- Example G – group, X — the set of finite subsets of G which contain e is a semilattice with respect to the union of subsets. $g * A$ is defined if $g^{-1} \in A$ in which case $g * A = gA = \{ga \colon a \in A\}$.

Partial action product

- Define $X \rtimes G = \mathcal{G}$, as a set.
- Let $\underset{x}{\bullet} \xrightarrow{g} \underset{y}{\bullet}$ and $\underset{x'}{\bullet} \xrightarrow{h} \underset{y'}{\bullet}$ be arrows.
- if $x' \neq y$ the product $(y', h) \cdot (y, g)$ is not defined in $\mathcal{G}$. Put $z = x' \wedge y$.
- Then $\underset{x''}{\bullet} \xrightarrow{g} \underset{z}{\bullet}$ and $\underset{z}{\bullet} \xrightarrow{h} \underset{y''}{\bullet}$ are in $\mathcal{G}$ where $x'' = g^{-1} * z$ and $y'' = h * z$.
- Put $(y', h) \circ (y, g) = (y'', h) \cdot (z, g) = (y'', hg)$.
- $\circ$ is called the pseudoproduct, $(X \rtimes G, \circ, ^{-1})$ is an inverse semigroup, which is E-unitary (see the next slide).

E-unitary inverse semigroups

- S – inverse semigroup, $E(S)$ – semilattice of idempotents of S.
- If γ is a group congruence on S (that is, S/γ is a group) then $e\,\gamma\,f$ for any $e, f \in E(S)$.
- If S contains 0 then γ is the universal congruence: $a = a \cdot a^{-1}a\,\gamma\,a \cdot 0 = 0$ for all $a \in S$.
- Let σ be the minimum group congruence. Then S/σ is the maximum group quotient of S. E.g.: if $S = \mathcal{I}(X)$ then $S/\sigma = \{0\}$.
- S is called E-unitary if $s\,\sigma\,e$ where $e \in E(S)$ implies that $s \in E(S)$.
- E.g.: groups and semilattices are E-unitary inverse semigroups.
- Let G be acting partially on X be order isomorphisms between order ideals. Then $X \rtimes G$ is called the *partial action product* of X by G. It is E-unitary.

Structure of E-unitary inverse semigroups

- S - inverse semigroup
- The underlying groupoid of S: vertices: $E(S)$, arrows $\underset{e}{\bullet} \xrightarrow{s} \underset{f}{\bullet}$ where $e = s^{-1}s =: \mathbf{d}(s)$, $f = ss^{-1} =: \mathbf{r}(s)$.
- Suppose S is E-unitary and $[s] = \sigma^{\natural}(s) \in S/\sigma =: G$.
- Then s is uniquely determined by $\mathbf{r}(s)$ (or $\mathbf{d}(s)$) and $[s]$.

  Indeed, let $s \ \sigma \ t$ and $ss^{-1} = tt^{-1}$. Then $st^{-1} \ \sigma \ tt^{-1} \Rightarrow st^{-1} \in E(S)$. So $ss^{-1}t \leq s$. By symmetry, $t \leq s$ so $t = s$.
- Let $g \in G$ and $e \in E(S)$. Put $g * e$ be defined if there is an arrow $\underset{e}{\bullet} \xrightarrow{s} \underset{f}{\bullet}$ with $[s] = g$ in the underlying groupoid of S in which case $g * e = f$. This is well defined and defines a *partial action* of G on $E(S)$ by order isomoprhisms between order ideals.
- Theorem (McAlister; interpretation by Kellendonk and Lawson) $S \simeq E(S) \rtimes G$.

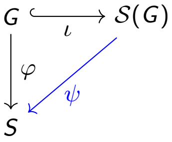
Partial actions of groups

and

actions of inverse semigroups

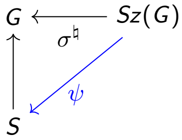
Exel's inverse semigroup $\mathcal{S}(G)$

- G – group, Exel's construction (1998), one of the (motivations) was to describe several classes of C^*-algebras which are cross products by partial actions of groups as cross products by actions of inverse semigroups.
- $\mathcal{S}(G)$ – universal semigroup given by generators $[g], g \in G$, and relations $[s][t][t^{-1}] = [st][t^{-1}]$, $[s^{-1}][s][t] = [s^{-1}][st]$, $[e]$ is the unit element. Then:
- $\mathcal{S}(G)$ is an inverse semigroup, and there is a bijection between partial actions of G and actions of $\mathcal{S}(G)$. Moreover:
- For any inverse semigroup S and any premorphism $\varphi\colon G \to S$ there is a unique morphism of semigroups $\psi\colon \mathcal{S}(G) \to S$ such that $\varphi = \psi\iota$.



$\mathcal{S}(G) \simeq Sz(G)$

- G – group, X — the set of finite subsets of G which contain e is a semilattice with respect to the union of subsets. $g * A$ is defined if $g^{-1} \in A$ in which case $g * A = gA = \{ga \colon a \in A\}$.
- $X \rtimes G =: Sz(G)$ – the Szendrei expansion of G (Szendrei, 1989).
- Fact: $X \rtimes G \simeq \mathcal{S}(G)$ (Kelendonk, Lawson, 2004).
- It is interesting that $Sz(G)$ has yet another universal property:
  1. First, $Sz(G)$ is an F-inverse monoid, that is, each σ-class has a maximum element. In addition $Sz(G)/\sigma \simeq G$ via the map $(A, g) \mapsto g$.
  2. F-inverse universal property. For any F-inverse monoid S and any F-inverse semigroup S with $S/\sigma \simeq G$, there is a unique morphism $\psi \colon Sz(G) \to S$ (which preserves maximum elements of σ-classes) such that the diagram below commutes:



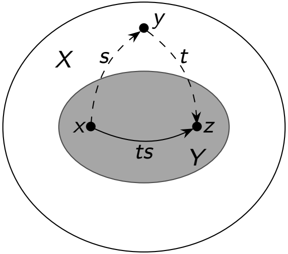
Partial representations of groups

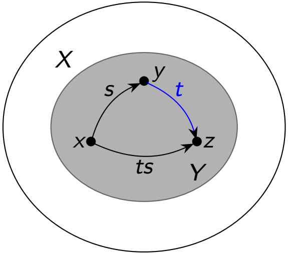
- A partial representation of a group G on a vector space V is a map $\varphi\colon G \to \mathrm{End}(V)$ such that $\varphi_e = \mathrm{id}_V$ and $\forall s, t \in S$: $\varphi_s\varphi_t\varphi_{t^{-1}} = \varphi_{st}\varphi_{t^{-1}}$, $\varphi_{s^{-1}}\varphi_s\varphi_t = \varphi_{s^{-1}}\varphi_{st}$.
- If K is a field, a partial group algebra $K_{par}(G)$ is the universal algebra given by generators $[s], s \in S$ and relations $[s][t][t^{-1}] = [st][t^{-1}]$, $[s^{-1}][s][t] = [s^{-1}][st]$, $s, t \in G$, $[e] = 1$.
- $K_{par}G \simeq K\mathcal{S}(G)$.
- Let $\Gamma(\mathcal{S}(G))$ be the underlying groupoid of $\mathcal{S}(G)$. The product of generators $s \cdot t$ is the product st in $\mathcal{S}(G)$ if $\mathbf{r}(t) = \mathbf{d}(s)$ and 0 otherwise.
- If G is finite then $K_{par}(G) \simeq K\Gamma(\mathcal{S}(G))$ (Dokuchaev, Exel, Piccione, 2000). Its dimension is $\sum_{k=1}^{n} \binom{n-1}{k-1} k = 2^{n-2}(n+1)$ (the cardinality of $\mathcal{S}(G)$).
- This result also follows from Steinberg (2006): $K(S) \simeq K\Gamma(S)$ for any inverse semigroup with finitely many idempotents.
- If S is infinite then a similar result holds with $\Gamma(S)$ replaced by the universal groupoid of S.

Partial actions of monoids

Partial actions of monoids which restrict actions

- M – monoid, X – set
- If M acts (globally) on X and $Y \subseteq X$. Let $*$ be the restricted partial action on Y.
- If $ts * x$ is defined, then $s * x$ does not need to be defined.
- If $ts * x$ and $s * x$ are defined then $t * (s * x)$ is defined and $ts * x = t * s * (x)$.





Partial groups actions without reference to inverses

- Recall:
- A partial action of G on X is a partial map $*: G \times X \to X, \ (g,x) \mapsto g*x$ (whenever defined) such that
  - (i) $e * x$ is defined and equals x for all $x \in X$
  - (ii) if $g * x$ is defined then $g^{-1} * (g * x)$ is defined and $g^{-1} * (g * x) = x$.
  - (iii) if $h * x$ and $g * (h * x)$ are defined then $gh * x$ is defined and $g * (h * x) = gh * x$.
- Observation (Megrelishvili, Schröder, 2004) In the definition above axiom (ii) can be replaced by
  - (iia) If $gh * x$ and $h * x$ are defined then $g * (h * x)$ is defined and $gh * x = g * h * (x)$.

Partial actions of monoids

- Definition. A partial action of M on X is a partial map $*: X \times M \to X$, $(x, s) \mapsto x * s$ (whenever defined) such that
  1. $e * x$ is defined and equals x for all $x \in X$.
  2. if $s * x$ and $t * (s * x)$ are defined then $ts * x$ is defined and $ts * x = t * (s * x)$.
- Definition A strong partial action of M on X is a partial action, which, in addition satisfies:
  3. If $gh * x$ and $h * x$ are defined then $g * (h * x)$ is defined and $gh * x = g * h * (x)$.
- Definition A premorphism $\varphi \colon M \to \mathcal{PT}(X)$[1]is a map such that $\varphi_e = \mathrm{id}_X$, $\varphi_s \varphi_t \leq \varphi_{st}$ for all $s, t \in M$. It is strong, if, in addition, $\varphi_s \varphi_t = \varphi_s^+ \varphi_{st}$.
- Every strong partial monoid action is globalizable (Megrelishvili and Schröder). It follows that strong partial monoid actions are precisely restrictions of actions.

[1] $\mathcal{PT}(X)$ is a left restriction monoid, instead of it one can consider any left restriction monoid.

Proper restriction semigroups and partial actions of monoids

Restriction semigroups

- Restrictions semigroups are non-regular generalizations of inverese semigroups. They have two unary operations $*$ and $+$. In an inverse semigroup $a^* = \mathbf{d}(a)$ and $a^+ = \mathbf{r}(a)$.
- E.g.: $R = \{f \in \mathcal{I}(X) \colon \forall x \in X \ f(x) \geq x\}$.
- More formally: restriction semigroups form a variety of algebras of signature $(\cdot, ^*, ^+)$, defined by the following identities:

$$x^+x = x, \ \ x^+y^+ = y^+x^+, \ \ (x^+y)^+ = x^+y^+, \ \ (xy)^+x = xy^+.$$

- Dual identites hold for $*$
- $(x^+)^* = x^+, (x^*)^+ = x^*$.
- $P(S) = \{x \in S \colon x = x^+ = x^*\}$ – semilattice of projection of S.
- Example Any monoid is a restriction semigroup with $x^* = x^+ = e$ for all x; as is any semilattice with $x^* = x^+ = x$.
- σ - minimum monoid congruence.
- *Aim:* generalize McAlister theorem to restriction semigroups. We need partial actions of monoids.

The partial action product

- Suppose that M acts partially on a semilattice X by order-isomorphisms between order ideals. Consider its underlying category.
- Note that an arrow $x \xrightarrow{s} y$ is uniquely determined by y and s only.
- Let $x \xrightarrow{s} y$ and $x' \xrightarrow{t} y'$ be arrows.
- Define $(y', t) \circ (y, s) = (y'', t) \cdot (z, s) = (y'', ts)$, where $z = x' \wedge y$, x'' is the source of the only arrow with label s and range y, and $y'' = t * z$.
- Define $(x, s)^+ = (x, e)$ and $(x, s)^* = (y, e)$ where $x = s * y$.
- $(X \rtimes G, \circ, ^+, ^*)$ is a restriction semigroup which is proper and every proper restriction semigroup arises this way (Cornock and Gould, 2011; GK, 2015)
- Proper means: $a^* = b^*$, $a \,\sigma\, b \Rightarrow a = b$ and $a^+ = b^+$, $a \,\sigma\, b \Rightarrow a = b$. Proper restriction semigroups generalize E-unitary inverse semigroups.
- This result has been extended to partial actions of restriction semigroups and to the structure of proper extensions of restirction semigroups (Dokuchaev, Khrypchenko, GK, 2021)

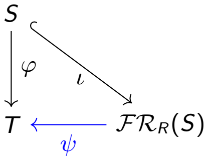
Almost perfect restriction semigroups

- M acts partially on X by partial bijections, and suppose that $\varphi\colon M \to \mathcal{I}(X)$, $m \mapsto \varphi_m$ is a morphism: $\varphi_s\varphi_t = \varphi_{st}$ holds.
- If G is a group $\varphi(G) \subseteq \mathcal{S}(X)$, so G acts on X. Consequently, $X \rtimes G$ is the semidirect product with respect to the action φ.
- If M is a monoid then the inclusion $\varphi(M) \subseteq \mathcal{S}(X)$ does not need to hold so we get a rich class of restriction semigroups, which does not have an adequate analogue if specialized to inverse semigroups.
- Partial action products with respect to homomorphisms are called almost perfect restriction semigroups (GK, 2015, called ultra proper, Jones 2016).
- The free restriction semigroup is almost perfect (but the free inverse semigroup is not).
- Every restriction semigroup has an almost perfect cover (which is not the case for inverse semigroups).
- Every left (or right) strong partial action of M on a semilattice by order-isomorphisms between order ideals is globalizable (GK, 2015; for inverse semigroups: Munn, 1976)

Expansions of monoids

$\mathcal{FR}_p(S)$ and $\mathcal{FR}_s(S)$

- S – a monoid, put $[S] = \{[s] \colon s \in S\}$.
- Define $\mathcal{FR}_p(S)$ and $\mathcal{FR}_s(S)$ to be the following restriction semigroups:
  1. $\mathcal{FR}_p(S) = \langle [S] \colon [e] = e, [s][t] \leq [st] \rangle$
  2. $\mathcal{FR}_s(S) = \langle [S] \colon [e] = e, [s][t] = [st][t]^* = [s]^+[st] \rangle$
- $\mathcal{FR}_s(S)$ is a generalization of $\mathcal{S}(G)$, $\mathcal{FR}_p(S)$ is a 'more relaxed' analogue of $\mathcal{S}(G)$.
- $\mathcal{FR}_p(S)$ and $\mathcal{FR}_s(S)$ are proper restriction semigroups, $\iota \colon S \to \mathcal{FR}_p(S)$ is a premorphism (resp. a strong premorphism).
- The universal property If $\varphi \colon S \to T$ is a premorphism to a restiction monoid then there is a morphism $\psi \colon \mathcal{FR}_p(S) \to T$ making the triangle commute. Similarly, for $\mathcal{FR}_s(S)$ and φ being strong.



The coordinatization

- In what follows R stands for one of p or s.
- Since $\mathcal{FR}_R(S)$ is proper, we have $\mathcal{FR}_R(S) \simeq P(\mathcal{FR}_R(S)) \rtimes S$.
- What is the structure of $P(\mathcal{FR}_R(S))$?
- Define $\mathcal{FI}_p(S)$ and $\mathcal{FI}_s(S)$ to be the following **inverse** semigroups:

  1. $\mathcal{FI}_p(S) = \langle [S] \colon [e] = e, [s][t] \leq [st] \rangle$.
  2. $\mathcal{FI}_s(S) = \langle [S] \colon [e] = e, [s][t] = [st][t]^* = [s]^+[st] \rangle$.

- Result (GK, 2019) $P(\mathcal{FR}_R(S)) \simeq E(\mathcal{FI}_R(S))$.
- S embeds into a group if and only if the canonical morphism $\mathcal{FR}_R(S) \to \mathcal{FI}_R(S)$ is injective.
- If S is an inverse monoid then $\mathcal{FI}_s(S)$ is isomorphic to the Lawson-Margolis-Steinberg generalized expansion.
- Corollary If the word problem in $\mathcal{FI}_R(S)$ is decidable, so is the word problem in $\mathcal{FR}_R(S)$.
- If M is finite then the word problem in $\mathcal{FR}_p(S)$ is decidable.

Globalization of partial actions of monoids and semigroups

The tensor product globalization

- S – a monoid, $*$ a strong partial action of S on X.
- $S \otimes X = S \otimes_S X = S \times X/\sim$, where $\sim$ is generated by $(ts, x) \sim (t, s * x)$.
- Define $t \circ (s \otimes x) = ts \otimes x$. This defines a global action of S on $S \otimes X$.
- Define $\delta\colon X \to S \otimes X$ by $x \mapsto e \otimes x$. It is an injection and if $s * x$ is defined, we have

$$s \circ (\delta(x)) = s \circ (e \otimes x) = s \otimes x = e \otimes s * x = \delta(s * x),$$

  so $(S \otimes X, \circ)$ is a globalization of $(X, *)$ via δ.
- A globalization $(Y, \cdot)$ of $(X, *)$ is X-generated (or an enveloping action), if $Y = S \cdot X$.
- $S \otimes X$ is a globalization of X (Hollings, 2007), which is an initial object in the category of all globalizations of X.
- If S is a group, $S \otimes X$ is, up to isomorphism, the only X-generated globalization of X (Kellendonk, Lawson, 2004).

Further results

- $*$ – partial action of a topological group G on a topological space X
- $G \star X = \{(g,x) \colon \exists g * x\}$; $*$ is continuous if the map $G \star X \to X$, $(g,x) \mapsto g * x$ is continuous
- Result (Kellendonk and Lawson, 2004; see also Abadie 2003) $*$ is globalizable if and only if:

  1. $G \star X$ is an open subset in $G \times X$ and
  2. $*$ is continuous.

  If $*$ is globalizable, then $G \otimes X$ is X-generated and is unique, up to homeomorphism.
- The unifying setting: globalization of geometric partial co(modules), see Saracco and Vercruysse, 2020, 2021.
- A partial group action by isomorphisms between ideals of an algebra is globalizable if and only if the domains of all φ_g are unital algebras, see Dokuchaev and Exel, 2004.
- Partial actions of groups on cell complexes were studied by Steinberg, 2003.

The Hom-set construction (GK and Laan, 2022)

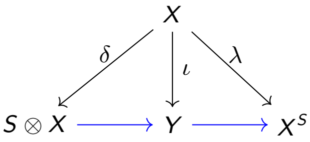
- Let $*$ – a partial action of a monoid S on a set X, $s \in S$ and $x \in X$. Put
$$\mathrm{dom}(f_{s,x}) = \{t \in S \colon ts * x \text{ is defined}\},$$
$$f_{s,x}(t) = ts * x \text{ for all } t \in \mathrm{dom}(f_{s,x}). \quad (1)$$
- Let
$$X^S = \{f_{s,x} \colon x \in X, s \in S\}.$$
- Define
$$t \circ f_{s,x} = f_{ts,x} \text{ for all } f_{s,x} \in X^S \text{ and } t \in S.$$
- Define $\lambda \colon X \to X^S$, $x \mapsto f_{e,x}$. It is an injection.
- Proposition. $(X^S, \circ)$ is an X-generated globalization of $(X, *)$ via λ.

An example: partially defined actions

- Let $\varphi\colon S \to \mathcal{PT}(X)$ be a homomorphism. We call it a partially defined action of S on X.
- Let us calculate X^S.
  - If $s * x$ is defined then $f_{s,x} = f_{e,s*x} \in \lambda(X)$.
  - If $s * x$ is undefined then $\operatorname{dom}(f_{s,x}) = \{t \in S \colon ts * x \text{ is defined}\} = \varnothing$, since $s * x$ is undefined implies that $ts * x$ is undefined for all $t \in T$. Define $f_{s,x} := o$.
  - It follows that $X^S = \lambda(X) \cup \{o\} := X \cup \{o\}$. We get the global S-act $(X \cup \{o\}, \circ)$ where
    $$s \circ x = \begin{cases} s * x, & \text{if } x \in X \text{ and } s * x \text{ is defined,} \\ o, & \text{otherwise.} \end{cases}$$
- So X^S is obtain via the well known embedding of $\mathcal{P}T(X)$ into $\mathcal{T}(X)$ by adding one new element to X.

The universal property

- S – monoid, $(X, *)$ – a strong partial S-act
- $\mathcal{G}_X(S, X, *)$ – the category of X-generated globalizations of $(X, *)$. Theorem $S \otimes X$ is an initial object and X^S is a terminal object in the category $\mathcal{G}_X(S, X, *)$.
- That is, if $(Y, \circ)$ is an X-generated globalization of $(X, *)$ via a map $\iota \colon X \to Y$ then there are unique morphisms of global S-acts $S \otimes X \to Y$, $s \otimes x \mapsto s * \iota(x)$, and $Y \to X^S$, $s * \iota(x) \mapsto f_{s,x}$, such that the following diagram commutes:

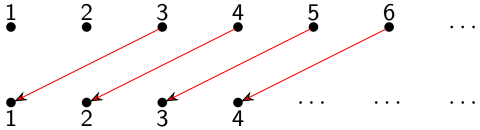


- The part about the terminal objects – GK and Laan (2022).

An example

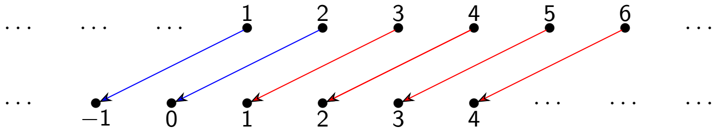
- $\mathbb{N}^0 = (\mathbb{N} \cup \{0\}, +)$ acts partially on $\mathbb{N}$ by setting $\varphi_n(a) = n \cdot a$ to be defined iff $a - n > 0$ in which case $n \cdot a = a - n$. Then $\cdot$ is a partially defined action.

The partially defined action of φ_2



- For $b \in B_{\mathcal{Z}} = \mathbb{Z}$ and $n \in \mathbb{N}^0$ put $\psi_n(b) = n * b = b - n$. Then $(B_{\mathcal{Z}}, *)$ is globalization of $(\mathbb{N}, \cdot)$ and is isomorphic to $\mathbb{N} \otimes \mathbb{N}^0$.

The action of ψ_2



An example: continuation

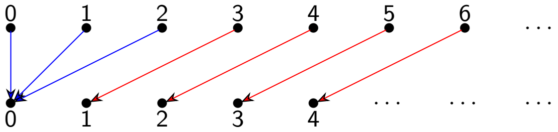
- For an integer $a \leq 0$ put $B_a = \{z \in \mathbb{Z} : z \geq a\}$. For each $b \in B_a$ and $n \in \mathbb{N}^0$ put

$$\gamma_n(b) = n *_a b = \begin{cases} b - n, & \text{if } b - n > a, \\ a, & \text{if } b - n \leq a. \end{cases}$$

  Then $(B_a, *_a)$ is an $\mathbb{N}$-generated globalization of $(\mathbb{N}, \cdot)$.

- If $a = 0$, $(B_0, *_0)$ is isomorphic to $\mathbb{N}^{\mathbb{N}^0}$.

$a = 0$, the action of γ_2



- All the constructed globalizations are pairwise non-isomorphic.

Partial actions of semigroups

- The constructions of $S \otimes X$ and X^S can be extended to suitable classes of partial actions of semigroups (which do not have a unit).
- One needs to restrict attention for classes of partial actions with a suitable substitution for the condition that $e * x$ is defined for all x and equals x.
- These classes are called firm and non-singular partial actions. Analogous classes for global actions arise naturally in Morita theory of semigroups.
- A partial action $\cdot$ of S on X is unital if for each $x \in X$ there are $y \in X$ and $s \in S$ such that $s \cdot y$ is defined and equals x.
- A partial action $\cdot$ of S on X is firm if it is unitary and whenever $s \cdot x$ and $t \cdot y$ are defined and $s \cdot x = t \cdot y$ we have $s \otimes x = t \otimes y$ in $S \otimes X$.
- The globalizations $S \otimes X$ and X^S are the initial object and the terminal object, respectively, in the appropriate categories of globalizations (GK, Laan, 2022).

Some questions and future work

- Study the Hom-set globalization in the case of action of a monoid on a topological space. Compare the results with those by Megrelishvili and Schröder.
- When is a partial monoid action by isomorphisms between ideals of an algebra globalizable? Can we construct the initial and the terminal product in the category of enveloping actions?
- Study partial cross products attached to partial actions of monoids (and semigroups) on algebras. Connect these with the universal category of a monoid.
- It is known that if $(\mathcal{B}, \beta)$ is an enveloping action of $(\mathcal{A}, \alpha)$ where a group G acts partially on a unital algebra $\mathcal{A}$, then the partial cross products $\mathcal{A} \rtimes G$ and $\mathcal{B} \rtimes G$ are Morita equivalent. Is there an analogue for G replaced by a monoid (or semigroup)?
- Similar question as above for the context of operator algebras.
- Fit the Hom-set globalization construction into the general framework of geometric partial (co)modules developed by Hu, Saracco and Vercruysse.