# Primeness of group-graded rings, with applications to partial crossed products and Leavitt path algebras

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# Outline

1. Background
   - Group rings
   - Strongly group-graded rings
   - Leavitt path algebras
2. Nearly epsilon-strongly graded rings
3. The main result
4. Comments on the proof
5. Applications
   - Leavitt path algebras
   - Partial crossed products
   - $s$-unital group rings

# Primeness of group rings

**Theorem (Connell, 1963)**

*Let $G$ be a group and let $R$ be a unital ring. TFAAE:*

1. (i) *The group ring $R[G]$ is prime.*
2. (ii) *$R$ is prime, and $G$ has no non-trivial finite normal subgroup.*

# Recall: Group-graded rings

From now on, $G$ denotes an arbitrary group.

### Definition

A ring $S$ is said to be *$G$-graded* if

- $S = \oplus_{g \in G} S_g$
- $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$.

### Definition

A $G$-graded ring $S$ is *strongly $G$-graded* if

- $S_g S_h = S_{gh}$ for all $g, h \in G$.

# Primeness of unital strongly graded rings

**Theorem (Passman, 1984)**

*Suppose that $S$ is a unital and strongly $G$-graded ring. Then $S$ is not prime if and only if there exist:*

1. *subgroups $N \lhd H \subseteq G$ with $N$ finite,*
2. *an $H$-invariant ideal $I$ of $S_e$ such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and*
3. *nonzero $H$-invariant ideals $\tilde{A}, \tilde{B}$ of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$.*

**Notation**

- $S_N := \bigoplus_{n \in N} S_n$
- $I^x := S_{x^{-1}} I S_x$

# The history of Leavitt path algebras

- 1962: Leavitt algebras
- 1977: Cuntz C*-algebras
- 1998: Graph C*-algebras
- 2005: Leavitt path algebras

Recommended survey article

G. Abrams,
*Leavitt path algebras: the first decade*,
Bull. Math. Sci. 5 (2015), no. 1, 59–120.

# Leavitt path algebras

### Definition

Let $E = (E^0, E^1)$ be a directed graph and let $R$ be a unital ring. The *Leavitt path algebra* $L_R(E)$ is the free associative $R$-algebra generated by the symbols $\{v \mid v \in E^0\} \cup \{f \mid f \in E^1\} \cup \{f^* \mid f \in E^1\}$ subject to the following relations:

a. $vw = \delta_{v,w} v$ for all $v, w \in E^0$;
b. $s(f)f = fr(f) = f$ and $r(f)f^* = f^* s(f) = f^*$, for every $f \in E^1$;
c. $f^* f' = \delta_{f,f'} r(f)$ for all $f, f' \in E^1$;
d. $\sum_{f \in E^1, s(f)=v} f f^* = v$ for every $v \in E^0$ for which $0 < |s^{-1}(v)| < \infty$.

We let every element of $R$ commute with the generators.

### Remark (A natural $\mathbb{Z}$-grading on $L_R(E)$)

Put: $\deg(v) = 0$, $\deg(f) = 1$, and $\deg(f^*) = -1$ for all $v$ and $f$.

# Leavitt path algebras are partial skew group algebras!

Theorem (Goncalves & Royer, 2014)

*Let $K$ be a field and let $E = (E^0, E^1)$ be a directed graph. Then $L_K(E) \cong D(X) \rtimes_\alpha \mathbb{F}$ as $K$-algebras.*

Explanation:

- $\mathbb{F}$ is the free group generated by $E^1$.
- $D(X)$ is a certain subalgebra of the function $K$-algebra on the set of sinks, infinite paths and finite paths ending in sinks.

# Primeness of Leavitt path algebras

### Definition

A directed graph $E$ is said to satisfy *condition (MT-3)* if for all $u, v \in E^0$, there exist $w \in E^0$ and paths from $u$ to $w$ and from $v$ to $w$.

### Theorem (Larki, 2015)

*Suppose that $E$ is a directed graph and that $R$ is a unital commutative ring. TFAAE:*

1. *The Leavitt path algebra $L_R(E)$ is prime.*
2. *$R$ is prime, and $E$ satisfies condition (MT-3).*

# Two key properties of unital strongly graded rings

## Definition

A $G$-graded ring $S$ is said to be *symmetrically $G$-graded* if

- $S_g S_{g^{-1}} S_g = S_g$ for every $g \in G$.

## Observation

Suppose that $S$ is a unital and strongly $G$-graded ring. Then:

- $S$ is symmetrically $G$-graded.
- $S_g S_{g^{-1}}$ is a unital ring for every $g \in G$ (because $S_g S_{g^{-1}} = S_e$).

# Epsilon-strongly graded rings

## Definition (Pinedo, Nystedt, Ö)

A $G$-graded ring $S$ is said to be *epsilon-strongly $G$-graded* if the following assertions hold:

- $S$ is symmetrically $G$-graded
- $S_g S_{g^{-1}}$ is a unital ring for every $g \in G$.

## Remark

An epsilon-strongly $G$-graded ring is always unital.

## Examples

- Every unital strongly $G$-graded ring.
- Every $\mathbb{Z}$-graded Leavitt path algebra $L_R(E)$, when $E$ is a finite graph.
- Every $G$-graded unital partial crossed product $R \rtimes_\alpha^w G$.

# Nearly epsilon-strongly graded rings

### Definition (Nystedt, Ö)

A $G$-graded ring $S$ is said to be *nearly epsilon-strongly $G$-graded* if the following assertions hold:

- $S$ is symmetrically $G$-graded
- $S_g S_{g^{-1}}$ is an $s$-unital ring for every $g \in G$.

### Examples

- Every epsilon-strongly $G$-graded ring.
- Every $\mathbb{Z}$-graded Leavitt path algebra $L_R(E)$, for any graph $E$.

### Observation (Lännström, 2021)

Every graded von Neumann regular ring is nearly epsilon-strongly graded.

# Yet another definition

## Definition

A $G$-graded ring $S$ is said to be *non-degenerately $G$-graded* if

- for every $g \in G$, and every nonzero $s \in S_g$, we have $sS_{g^{-1}} \neq \{0\}$ and $S_{g^{-1}}s \neq \{0\}$.

## Remark

A nearly epsilon-strongly $G$-graded ring $S$ is non-degenerately $G$-graded.

## Proof (50%).

- Take $s \in S_g$ and suppose that $S_{g^{-1}}s = \{0\}$.
- Then $s = \sum_{i=1}^{n} a_i b_i c_i$ where $a_i, c_i \in S_g$ and $b_i \in S_{g^{-1}}$.
- Let $u \in S_g S_{g^{-1}}$ be an $s$-unit for $\{a_1 b_1, \ldots, a_n b_n\}$.
- $s = \sum_{i=1}^{n} a_i b_i c_i = \sum_{i=1}^{n} (u a_i b_i) c_i = us \in S_g S_{g^{-1}} s = \{0\}$.

□

## Theorem (Lännström, Lundström, Ö, Wagner)

*Suppose that $G$ is a group and that $S$ is a $G$-graded ring. Consider the following five assertions:*

- (a) *$S$ is not prime.*
- (b) *There exist:*
  - (i) *subgroups $N \lhd H \subseteq G$ with $N$ finite,*
  - (ii) *an $H$-invariant ideal $I$ of $S_e$ such that $I^x I = \{0\}$ for every $x \in G \setminus H$,*
  - (iii) *nonzero ideals $\tilde{A}, \tilde{B}$ of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A} S_H \tilde{B} = \{0\}$.*
- (c) *There exist:*
  - (i) *subgroups $N \lhd H \subseteq G$ with $N$ finite,*
  - (ii) *an $H$-invariant ideal $I$ of $S_e$ such that $I^x I = \{0\}$ for every $x \in G \setminus H$,*
  - (iii) *nonzero $H$-invariant ideals $\tilde{A}, \tilde{B}$ of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A} S_H \tilde{B} = \{0\}$.*

*The following assertions hold:*

1. *If $S$ is non-degenerately $G$-graded, then (c)$\Longrightarrow$(b)$\Longrightarrow$(a).*
2. *If $S$ is nearly epsilon-strongly $G$-graded, then (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c).*

# The "easy" direction

## Proposition

*Suppose that $S$ is non-degenerately $G$-graded and that there exist*

1. *subgroups $N \triangleleft H \subseteq G$ with $N$ finite,*
2. *an $H$-invariant ideal $I$ of $S_e$ such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and*
3. *nonzero ideals $\tilde{A}, \tilde{B}$ of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$, and $\tilde{A} S_H \tilde{B} = \{0\}$.*

*Then $S$ is not prime.*

## Proof (sketch).

Consider the ideals $A := S\tilde{A}S$ and $B := S\tilde{B}S$ of $S$.

- By non-degeneracy of the grading, $A$ and $B$ are both nonzero.
- One can show that $\tilde{A} S_g \tilde{B} = \{0\}$ for every $g \in G$. From this we get that $AB = \{0\}$.

□

# The "hard" direction

**Definition**

Let $S$ be a $G$-graded ring. An *NP-datum* for $S$ is a quintuple $(H, N, I, \tilde{A}, \tilde{B})$ with the following three properties:

(NP1) $H$ is a subgroup of $G$, and $N$ is a finite normal subgroup of $H$,

(NP2) $I$ is a nonzero $H$-invariant ideal of $S_e$ such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and

(NP3) $\tilde{A}, \tilde{B}$ are nonzero ideals of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$, and $\tilde{A}\tilde{B} = \{0\}$.

An NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ is said to be *balanced* if it satisfies the following property:

(NP4) $\tilde{A}, \tilde{B}$ are nonzero ideals of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$, and $\tilde{A} S_H \tilde{B} = \{0\}$.

# The "hard" direction

**Remark**

If $S$ is nearly epsilon-strongly $G$-graded, then (NP4) implies (NP3).

**Remark**

Suppose that $S$ is $s$-unital strongly $G$-graded. An NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for $S$ is necessarily balanced whenever $\tilde{A}$ or $\tilde{B}$ is $H$-invariant. Indeed, suppose that $\tilde{A}$ is $H$-invariant. For any $h \in H$, we get that

$$\tilde{A}S_h\tilde{B} = S_e\tilde{A}S_h\tilde{B} = S_hS_{h^{-1}}\tilde{A}S_h\tilde{B} \subseteq S_h\tilde{A}\tilde{B} = \{0\}.$$

The proof of the case when $\tilde{B}$ is $H$-invariant is analogous.

# The "hard" direction

**Proposition**

*Suppose that $S_e$ is not $G$-prime. Then $S$ has a balanced NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for which $\tilde{A}, \tilde{B}$ are $H$-invariant.*

**Proof.**

- There are nonzero $G$-invariant ideals $\tilde{A}, \tilde{B}$ of $S_e$ such that $\tilde{A}\tilde{B} = \{0\}$.
- We claim that $(G, \{e\}, S_e, \tilde{A}, \tilde{B})$ is a balanced NP-datum.
- Conditions (NP1), (NP2) and (NP3) are trivially satisfied.
- We now check condition (NP4).
  Take $x \in G$. Seeking a contradiction, suppose that $\tilde{A}S_x\tilde{B} \neq \{0\}$. Note that $\tilde{A}S_x\tilde{B} \subseteq S_x$. By non-degeneracy of the $G$-grading, $S_{x^{-1}} \cdot \tilde{A}S_x\tilde{B} \neq \{0\}$. Since $\tilde{A}$ is $G$-invariant, we get that $S_{x^{-1}}\tilde{A}S_x\tilde{B} \subseteq \tilde{A}\tilde{B} = \{0\}$, which is a contradiction. Note that, trivially, $\tilde{A}, \tilde{B}$ are both $G$-invariant.

# The "hard" direction

**Proposition**

*Suppose that $S$ is nearly epsilon-strongly $G$-graded. If $S$ is not prime, then it has a balanced NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for which $\tilde{A}, \tilde{B}$ are $H$-invariant.*

**Comment on the proof.**

Case 1 ($S_e$ is not $G$-semiprime): Previous slide.
Case 2 ($S_e$ is $G$-semiprime): Requires long ($\approx$ 15 pages) and technical arguments... □

# Primeness of Leavitt path algebras

**Theorem (Lännström, Lundström, Ö, Wagner)**

*Suppose that $E$ is a directed graph and that $R$ is a unital ring. TFAAE:*

1. *The Leavitt path algebra $L_R(E)$ is prime.*
2. *$R$ is prime, and $E$ satisfies condition (MT-3).*

# Primeness of partial crossed products

**Remark**

Let $R \star_\alpha^w G$ be a unital partial crossed product coming from a unital twisted partial action $(\{\alpha_g\}_{g\in G}, \{D_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G})$.

- An ideal $I$ of $R$ is *$G$-invariant* if $\alpha_g(I \cap D_{g^{-1}}) \subseteq I$ for every $g \in G$.
- $R$ is *$G$-prime* if there are no nonzero $G$-invariant ideals $I, J$ of $R$ such that $IJ = \{0\}$.

**Theorem (Lännström, Lundström, Ö, Wagner)**

*Suppose that $G$ is torsion-free and that $R \star_\alpha^w G$ is a unital partial crossed product. Then $R \star_\alpha^w G$ is prime if and only if $R$ is $G$-prime.*

# The $s$-unital Connell's theorem

Let $R$ be an $s$-unital ring. We define the *group ring* $R[G]$ as the set of all formal sums $\sum_{x \in G} r_x \delta_x$ where $\delta_x$ is a symbol for each $x \in G$ and $r_x \in R$ is zero for all but finitely many $x \in G$. Addition on $R[G]$ is defined in the natural way and multiplication is defined by linearly extending the rules $r\delta_x r'\delta_y = rr'\delta_{xy}$, for all $r, r' \in R$ and $x, y \in G$.

Theorem (Lännström, Lundström, Ö, Wagner)

*Let $R$ be an $s$-unital ring. TFAAE:*

1. *The group ring $R[G]$ is prime.*
2. *$R$ is prime, and $G$ has no non-trivial finite normal subgroup.*

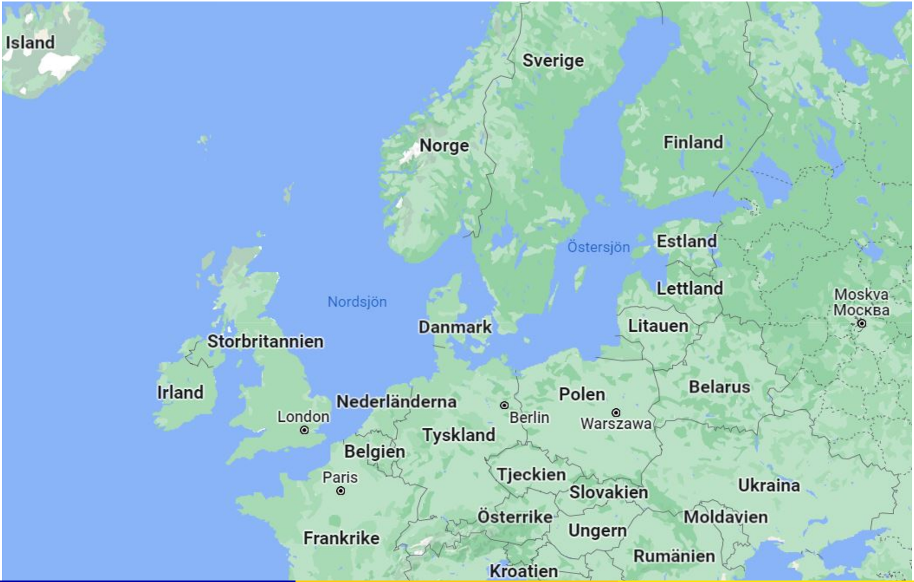
# Reference

D. Lännström, P. Lundström, J. Öinert and S. Wagner,

*Prime group graded rings with applications to partial crossed products and Leavitt path algebras*,

arXiv:2105.09224 [math.RA]

# The location of Blekinge Institute of Technology



# The location of Blekinge Institute of Technology

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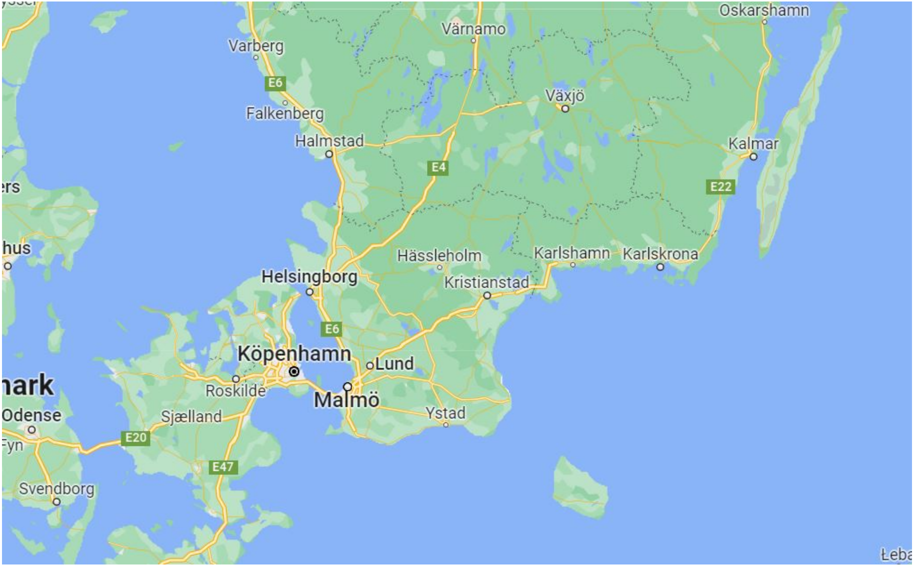

# The campus in Karlskrona

# The end

THANK YOU FOR YOUR ATTENTION!