Hopf25

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Simple algebras in $\mathrm{Rep}(u_q(\mathfrak{sl}_2))$

This talk is based on our joint work with Daisuke Nakamura, Hin Wang Ng and Taiki Shibata. In the theory of finite tensor categories and their applications, the notion of an algebra in a finite tensor category often plays an important role. As in the ordinary ring theory, simple algebras are one of the most fundamental classes of algebras. Coulembier, Stroiński, and Zorman recently showed that the category $\mathcal{C}_A$ of right modules over a simple algebra A in a finite tensor category $\mathcal{C}$ is an indecomposable exact module category over $\mathcal{C}$, as conjectured by Etingof and Ostrik. Given this result and others on finite tensor categories and their modules, we are interested in studying and classifying simple algebras with some properties or structures.

A typical example of a finite tensor category is the category $\mathrm{Rep}(H)$ of representations of a finite-dimensional Hopf algebra H. In this talk, after reviewing basic results on simple algebras in finite tensor categories, I will present the progress of our project to classify simple algebras in $\mathrm{Rep}(u_q(\mathfrak{sl}_2))$ with some properties or structures, where $u_q(\mathfrak{sl}_2)$ is the small quantum group associated with $\mathfrak{sl}_2$ at a root of unity q of odd order. Let, in general, $\mathcal{C}$ be a finite tensor category. It is known that every simple algebra in $\mathcal{C}$ is of the form $\underline{\mathrm{End}}_{\mathcal{M}}(X)$, where X is a non-zero object of an indecomposable exact module category $\mathcal{M}$ over $\mathcal{C}$ and $\underline{\mathrm{End}}_{\mathcal{M}}$ means the internal endomorphism algebra in $\mathcal{C}$. A Morita theoretic argument shows that $A := \underline{\mathrm{End}}_{\mathcal{M}}(X)$ is simple in $\mathcal{C}_A$ if and only if X is simple. Using the relative Serre functor of $\mathcal{M}$, one can also characterize when A is (symmetric) Frobenius. Given an object Y of another indecomposable exact module category $\mathcal{N}$ over $\mathcal{C}$, the algebras $\underline{\mathrm{End}}_{\mathcal{N}}(Y)$ and $\underline{\mathrm{End}}_{\mathcal{M}}(X)$ are isomorphic if and only if there is an equivalence $\mathcal{M} \to \mathcal{N}$ of module categories sending X to Y.

By applying the above abstract results for $\mathcal{C} = \mathrm{Rep}(u_q(\mathfrak{sl}_2))$, we obtain a list of simple algebras in $\mathrm{Rep}(u_q(\mathfrak{sl}_2))$ that is simple as a right module, in the form of an internal endomorphism algebra, and determine whether it is (symmetric) Frobenius.

Since the computation of the internal endomorphism algebra is not an easy problem in general, our list of simple algebras is still somewhat implicit. I will introduce some techniques or attempts to find generators and relations of the internal endomorphism algebra. I will also discuss the braided commutativity of simple algebras in $\mathrm{Rep}(u_q(\mathfrak{sl}_2))$.