On the finite generation of the cohomology of abelian extensions of Hopf algebras

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I. Antecedents. Let $\Bbbk$ be a field; we may assume that $\Bbbk = \overline{\Bbbk}$. Let G be a finite group and $H = \Bbbk G$ the group algebra.

Theorem. (Maschke, 1898)
If $\operatorname{char} \Bbbk \nmid |G|$, then $H = \Bbbk G$ is semisimple.

When $\operatorname{char} \Bbbk \mid |G|$, H is not semisimple and one would need to compute the groups

$$\operatorname{Ext}^n_H(N, M)$$

for any finitely generated H-modules N and M, any $n \in \mathbb{N}$. Note:

$$\operatorname{Ext}^n_H(\Bbbk, M \otimes N^*) \simeq \operatorname{Ext}^n_H(N, M).$$

Theorem (Golod, 1959; Evens, 1961; Venkov, 1959).
The following properties hold:

(fgc-i) The cohomology ring

$$\mathtt{H}(H, \Bbbk) = \mathrm{Ext}_H^\bullet(\Bbbk, \Bbbk) = \bigoplus_{n \in \mathbb{N}_0} \mathrm{Ext}_H^n(\Bbbk, \Bbbk)$$

is finitely generated, and

(fgc-ii) For any finitely generated H-module M,

$$\mathtt{H}(H, M) = \mathrm{Ext}_H^\bullet(\Bbbk, M) = \bigoplus_{n \in \mathbb{N}_0} \mathrm{Ext}_H^n(\Bbbk, M)$$

is a finitely generated $\mathtt{H}(H, \Bbbk)$-module.

Theory of the support variety (Quillen, 1971):

Use algebraic geometry to study the representation theory of G, more precisely via $\mathrm{Ext}^{\bullet}_{H}(\Bbbk, \Bbbk)$ and the support of $\mathrm{Ext}^{\bullet}_{H}(\Bbbk, M)$.

If $\operatorname{char} \mathbb{k} = 0$, any cocommutative finite-dimensional Hopf algebra is a group algebra.

If $\operatorname{char} \mathbb{k} > 0$, there are more cocommutative finite-dimensional Hopf algebras (difficult to classify).

Example. The restricted enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of a fin.-dim. restricted Lie algebra $\mathfrak{g}$ is a fin.-dim. cocommutative Hopf alg.

Theorem (Friedlander & Parshall, 1983).
Let $H = \mathfrak{u}(\mathfrak{g})$, $\mathfrak{g}$ a finite-dimensional restricted Lie algebra.

(fgc-i) The cohomology ring $\mathsf{H}(H, \mathbb{k})$ is finitely generated.

(fgc-ii) For any finitely generated H-module M, $\mathsf{H}(H, M)$ is a finitely generated module over $\mathsf{H}(H, \mathbb{k})$.

Assume that $\operatorname{char} \mathbb{k} = p > 0$.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. The nilpotent cone of $\mathfrak{g}$ is $\mathcal{N} = \{x \in \mathfrak{g} : (\operatorname{ad} x)^n = 0,\ n \gg 0\}$.

Let $\mathfrak{G}$ be the associated restricted Lie algebra over $\mathbb{k}$, $H = \mathfrak{u}(\mathfrak{G})$.

Theorem (Friedlander & Parshall, 1986).
The cohomology ring $\mathrm{H}(H, \mathbb{k})$ is isomorphic to the graded ring of polynomial functions over $\mathcal{N}$.

Remark. These three categories are equivalent:

$$\{\text{cocommutative finite-dimensional Hopf algebras}\},$$
$$\{\text{commutative finite-dimensional Hopf algebras}\}^{\text{op}},$$
$$\{\text{finite group schemes}\}.$$

Theorem (Friedlander & Suslin, 1997).
Let H be a cocommutative finite-dimensional Hopf algebra (that is, a finite group scheme).

(fgc-i) The cohomology ring $\mathrm{H}(H, \Bbbk)$ is finitely generated.

(fgc-ii) For any finitely generated H-module M, $\mathrm{H}(H, M)$ is a finitely generated module over $\mathrm{H}(H, \Bbbk)$.

In the same paper, Friedlander & Suslin observe that the cohomology ring of a finite-dimensional *commutative* Hopf algebra is easily seen to be finitely generated using the structure and add:

We do not know whether it is reasonable to expect finite generation of the cohomology of an arbitrary finite-dimensional Hopf algebra.

Definition. We say that a finite-dimensional augmented algebra H has finite generation of the cohomology **(fgc)** if
(fgc-i) The cohomology ring $\mathrm{H}(H, \Bbbk)$ is finitely generated.

(fgc-ii) For any finitely generated H-module M, $\mathrm{H}(H, M)$ is a finitely generated module over $\mathrm{H}(H, \Bbbk)$.

Theorem (Ginzburg & Kumar 1993).
(char $\mathbb{k} = 0$). Let H be the Frobenius-Lusztig kernel (aka small quantum group) $\mathfrak{u}_q(\mathfrak{g})$, $\mathfrak{g}$ a simple Lie algebra, $q \in \mathbb{G}_\infty$ with restrictions on the order.
Then H has **(fgc)**.

Actually they prove that $\mathsf{H}(H, \mathbb{k})$ **is isomorphic to the algebra of rational functions on the nilpotent cone of** $\mathfrak{g}$**.**

Note: Similar, more restricted result by Verbistky & Kazhdan.

Conjecture. (Etingof & Ostrik, 2005) A finite tensor category $\mathcal{C}$ (e.g. $\mathcal{C} = \operatorname{rep} H$, H finite-dimensional Hopf algebra) has fgc:

(fgc-i) The cohomology ring $\operatorname{Ext}^\bullet_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ is finitely generated.

(fgc-ii) If $M \in \mathcal{C}$, $\operatorname{Ext}^\bullet_{\mathcal{C}}(\mathbf{1}, M)$ is a fin. gen. $\operatorname{Ext}^\bullet_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$-module.

Known in many cases, for instance:

- (Gordon 2000). (char $\Bbbk = 0$). $H = \mathfrak{u}_q(\mathfrak{g})^*$, $\mathfrak{g}$ a simple Lie algebra, $q \in \mathbb{G}_\infty$ with restrictions on the order, has **(fgc)**.

- (Drupieski 2011). (char $\Bbbk > 0$). $H = \mathfrak{u}_q(\mathfrak{g})$, $\mathfrak{g}$ a simple Lie algebra, $q \in \mathbb{G}_\infty$ with restrictions on the order, has **(fgc)**.

- (Drupieski 2016). char $\Bbbk > 0$: Finite supergroup schemes have **(fgc)**.

- (Mastnak-Petvsova-Schauenburg-Witherspoon 2010) (char $\Bbbk = 0$).
H pointed, $G(H)$ abelian, $(|G(H)|, 210) = 1$, has **(fgc)**..

- (A-Angiono-Petvsova-Witherspoon 2022) (char $\Bbbk = 0$).
If H is pointed, $G(H)$ abelian and the associated Nichols algebra comes in families, then $D(H)$ has **(fgc)**.

- (Stefan & Vay 2016). (char $\Bbbk = 0$).
$H = \mathscr{B}(V)\#\Bbbk\mathbb{S}_3$, where $\mathscr{B}(V) \simeq \mathtt{FK}_3$, $\dim \mathscr{B}(V) = 12$, has **(fgc)**.

- (Nguyen, Wang & Witherspoon; Erdmann, Solberg & Wang 2018).
(char $\Bbbk = p > 0$). (Some) pointed Hopf algebras of dim p^3 have **(fgc)**.

- (Friedlander & Negron, 2019; Negron, 2021) If H is cocommutative, then $D(H)$ has **(fgc)**.

II. Morita equivalence.

Let $H = (H, m, \Delta)$ be a finite-dim. Hopf algebra.

- $H^* = (H, \Delta^t, m^t)$
- $H^F = (H, m, \Delta^F)$
- $H_\sigma = (H, m_\sigma, \Delta)$
- $D(H) =$ Drinfeld double

Questions:

Let $H = (H, m, \Delta)$ be a finite-dim. Hopf algebra.
If H has **(fgc)**, does...

- $H^* = (H, \Delta^t, m^t)$ have **(fgc)?**
- $H^F = (H, m, \Delta^F)$ have **(fgc)?**
- $H_\sigma = (H, m_\sigma, \Delta)$ have **(fgc)?**
- $D(H) =$ Drinfeld double have **(fgc)?**

Let H and U be finite-dimensional Hopf algebras.

We say that H and U are *Morita equivalent*, $H \sim_{\text{Mor}} U$, if there exists an equivalence of braided tensor categories between the Drinfeld centers $\mathcal{Z}(\text{rep}\, H)$ and $\mathcal{Z}(\text{rep}\, U)$ (Müger, Etingof-Nikshych-Ostrik). Equivalently, $D(H) \simeq D(U)^G$, G a twist.

Example. $H \sim_{\text{Mor}} H^*$, $H \sim_{\text{Mor}} H^F$, $H \sim_{\text{Mor}} H_\sigma$.

Remark. This is not the same as Morita equivalence of algebras.

Lemma. [AAPW, Negron-Plavnik] Let R be an augmented subalgebra of a finite-dimensional augmented algebra A.
Suppose that A is projective as a right R-module under multiplication. If A has fgc, then so does R.

Corollary. [AAPW, Negron-Plavnik] Let H and U be finite-dim. Morita equivalent Hopf algebras. If $D(H)$ has fgc, then U has fgc.

Indeed, $D(H)$ has fgc $\Longrightarrow$ $D(U)$ has fgc $\Longrightarrow$ U has fgc.

In particular, if $D(H)$ has fgc, then H^*, H^F, H_σ have fgc.

Scheme of the proof of Theorem [AAPW].

Theorem A. Let U be a braided vector space of diagonal type such that the Nichols algebra $\mathscr{B}(U)$ has finite dimension. Then $\mathscr{B}(U)$ has fgc.

The rest of the proof: Let H be a f.d. pointed Hopf algebra with $G(H)$ abelian & infinitesimal braiding V, so that $H \sim_{\text{Morita}} \operatorname{gr} H \simeq \mathscr{B}(V)\#\Bbbk G(H)$ (Angiono, Angiono-García Iglesias). Then

$$\begin{array}{ccccc}
\text{Theorem A} & \Longrightarrow & \mathscr{B}(V),\ \mathscr{B}(V^*) \text{ have fgc} & \Longrightarrow & \operatorname{gr} H,\ (\operatorname{gr} H)^* \text{ have fgc} \\
 & & & & \Downarrow \\
H \text{ has fgc} & \Longleftarrow & D(H) \text{ has fgc} & \Longleftrightarrow & D(\operatorname{gr} H) \text{ has fgc}
\end{array}$$

III. Extensions.

Consider an extension of finite-dimensional Hopf algebras:

$$\Bbbk \to K \to H \to L \to \Bbbk \qquad (*)$$

As we have seen, If H has fgc, then so does K.

Question.[A-Natale] If H has fgc, does L also have fgc?

Lemma. [AAPW] K semisimple, L has fgc, then so does H.

Question. [A-Natale] If K and L have fgc, does H also have fgc?

Consider an *abelian* extension of finite-dim. Hopf algebras:

$$\Bbbk \to K \to H \to L \to \Bbbk, \qquad (*)$$

that is, K is commutative and L is cocommutative.
Thus, K and L have fgc (and H if char $\Bbbk = 0$; assume char $\Bbbk > 0$).

Split extensions. [G. I. Kac, Majid] The following are equivalent:

exact factorizations $\leftrightarrow$ matched pairs $\leftrightarrow$ split extensions

$$S = G \cdot L \leftrightsquigarrow (G, L, \triangleright, \triangleleft) \leftrightsquigarrow (L^*, G, \leftharpoonup, \rho)$$

$$S = G \bowtie L \longleftarrow (G, L, \triangleright, \triangleleft) \leftrightsquigarrow L^* \hookrightarrow L^* \# G \twoheadrightarrow G.$$

Note: S cocommutative $\Leftrightarrow L^* \hookrightarrow L^* \# G \twoheadrightarrow G$ abelian extension.

Note: Any abelian extension is like $L^* \hookrightarrow L^{*\tau} \#_\sigma G \twoheadrightarrow G$
for suitable 2-cocycles τ and σ.

Consider an *abelian* extension of finite-dim. Hopf algebras:

$$\Bbbk \to K \to H \to L \to \Bbbk, \qquad (*)$$

Definition. We say that H is *quasi-split* if it is Morita equivalent to the split extension: $H \sim_{\text{Mor}} K\#L$.

Theorem. (Schauenburg) If $(*)$ is a split abelian extension, then there is a cocommutative Hopf algebra U such that $H \sim_{\text{Mor}} U$ (actually, $U \simeq L \bowtie K^*$).

Theorem. (A.-Natale) If H is a *quasi-split* abelian extension, then $D(H)$, hence H and any Hopf algebra $U \sim_{\text{Mor}} H$, have fgc.

Proof: Negron + Schauenburg.

IV. Applications. Assume in this Section that $\operatorname{char} \Bbbk > 2$.

A class of braided vector spaces $\mathscr{V}$ was introduced in (A-Angiono-Heckenberger); they decompose as direct sums of Jordan blocks, super Jordan blocks and labelled points. Their Nichols algebras are finite-dimensional.

Let $\mathscr{V}_+$ be the subclass with only Jordan blocks and points labelled with 1; these depend on a family of parameters:

$$\Lambda \ni (\mathfrak{q}, \mathbf{a}) \rightsquigarrow \mathscr{V}(\mathfrak{q}, \mathbf{a}).$$

Let $(\mathfrak{q}, \mathbf{a}) \in \Lambda$.

Theorem. (A.-Natale) For a suitable finite abelian group Γ,

• the bosonization $H = \mathscr{B}(\mathscr{V}(\mathbf{1}, \mathbf{a}))\#\Bbbk\Gamma$ fits into a *split* abelian exact sequence, hence $D(H)$, H and $\mathscr{B}(\mathscr{V}(\mathbf{1}, \mathbf{a}))$ have fgc;

• for a general $\mathfrak{q}$, $\mathscr{B}(\mathscr{V}(\mathfrak{q}, \mathbf{a}))\#\Bbbk\Gamma$ is a cocycle deformation of H, hence $D(\mathscr{B}(\mathscr{V}(\mathfrak{q}, \mathbf{a}))\#\Bbbk\Gamma)$, $\mathscr{B}(\mathscr{V}(\mathfrak{q}, \mathbf{a}))\#\Bbbk\Gamma$ and $\mathscr{B}(\mathscr{V}(\mathfrak{q}, \mathbf{a}))$ have fgc too.

As illustration, we give details for two simple examples.

The *Jordan block* $\mathcal{V}(1,2)$ is the braided vector space with basis $\{x, y\}$ such that

$$c(x \otimes x) = x \otimes x, \qquad c(y \otimes x) = x \otimes y,$$
$$c(x \otimes y) = (y + x) \otimes x, \qquad c(y \otimes y) = (y + x) \otimes y.$$

Lemma. [Cibils-Lauve-Witherspoon]
The Nichols algebra $\mathscr{B}(\mathcal{V}(1,2))$ (called the *restricted Jordan plane*) is generated by x, y with relations

$$yx - xy + \tfrac{1}{2}x^2, \qquad x^p, \qquad y^p.$$

$\{x^a y^b : 0 \leq a, b < p\}$ is a basis of $\mathscr{B}(\mathcal{V}(1,2)) \Rightarrow \dim \mathscr{B}(\mathcal{V}(1,2)) = p^2$.

The minimal bosonization.

Let $\Gamma = \mathbb{Z}/p = \langle g \rangle$. We realize $\mathcal{V}(1,2)$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by

$$g \cdot x = x, \qquad g \cdot y = y + x, \qquad \deg x = \deg y = g.$$

Thus the Hopf algebra $H = \mathscr{B}(\mathcal{V}(1,2))\#\Bbbk\Gamma$ has dimension p^3. Let $K = \Bbbk\langle x, g\rangle \subset H$ and $L = \Bbbk[\zeta]/(\zeta^p)$ with ζ primitive.

Lemma. [A-Natale] H fits into a split abelian extension

$$\Bbbk \to K \xrightarrow{\iota} H \xrightarrow{\pi} L \to \Bbbk,$$

ι is the inclusion & π is defined by $\pi(x) = 0$, $\pi(g) = 1$, $\pi(y) = \zeta$.

Remark. [A-Peña Pollastri] The Drinfeld double of H fits into an abelian exact sequence $\Bbbk \to \mathbf{R} \to D(H) \to \mathfrak{u}(\mathfrak{sl}_2(\Bbbk)) \to \Bbbk$, where $\mathbf{R}$ is a local commutative Hopf algebra.

Proposition.
(i) [Nguyen-Wang-Witherspoon] The Hopf algebra $\mathscr{B}(\mathcal{V}(1,2))\#\Bbbk\Gamma$ and the Nichols algebra $\mathscr{B}(\mathcal{V}(1,2))$ have fgc.

(ii) [A-Natale] The Drinfeld double $D\left(\mathscr{B}(\mathcal{V}(1,2))\#\Bbbk\Gamma\right)$ has fgc.

Question: Let F be another finite group such that $\mathcal{V}(1,2)$ admits a realization in ${}^{\Bbbk F}_{\Bbbk F}\mathcal{YD}$. Does $\mathscr{B}(\mathcal{V}(1,2))\#\Bbbk F$ have fgc?

Proposition. [A-Natale] $(\mathscr{B}(\mathcal{V}(1,2))\#\Bbbk F)^*$ has fgc.

The Nichols algebra $\mathscr{B}(\mathfrak{L}_q(1,\mathscr{G}))$. Let $q \in \Bbbk^\times$, $a \in \mathbb{F}_p^\times$ and

$$\mathtt{r} \in \{1-p, 2-p, \dots, -2, -1\} \quad \text{such that} \quad \mathtt{r} \equiv 2a \mod p.$$

The *ghost* is $\mathscr{G} := -\mathtt{r} \in \{1, \dots, p-1\}$; since p is odd, $\mathscr{G}$ gives a.

The braided vector space $\mathfrak{L}_q(1,\mathscr{G})$ has basis $\mathscr{b} = \{x_1, y_1, x_2\}$ and

$$(c(b \otimes b'))_{b,b' \in \mathscr{b}} = \begin{pmatrix} x_1 \otimes x_1 & (y_1 + x_1) \otimes x_1 & q\, x_2 \otimes x_1 \\ x_1 \otimes y_1 & (y_1 + x_1) \otimes y_1 & q\, x_2 \otimes y_1 \\ q^{-1} x_1 \otimes x_2 & q^{-1}(y_1 + a x_1) \otimes x_2 & x_2 \otimes x_2 \end{pmatrix}.$$

Thus $V_1 := \Bbbk x_1 + \Bbbk y_1 \simeq \mathcal{V}(1,2)$ and $V_2 := \Bbbk x_2$ satisfy

$$c : V_i \otimes V_j = V_j \otimes V_i, \qquad i, j \in \{1, 2\}.$$

Hence V_1 and V_2 are braided subspaces of V.

Set $z_0 := x_2$, $z_{n+1} := y_1 z_n - q z_n y_1$, $n > 0$.

Lemma. [A-Angiono-Heckenberger] The Nichols algebra $\mathscr{B}(\mathfrak{L}_q(1, \mathscr{G}))$ is generated by x_1, y_1, x_2 with relations

$$\begin{aligned} & y_1 x_1 - x_1 y_1 + \tfrac{1}{2} x_1^2, && x_1^p, \; y_1^p. \\ x_1 x_2 &= q\, x_2 x_1, \\ z_{1+\mathscr{G}} &= 0, \\ z_t z_{t+1} &= q^{-1}\, z_{t+1} z_t, && 0 \le t < \mathscr{G}, \\ z_t^p &= 0, && 0 \le t \le \mathscr{G}. \end{aligned}$$

It has $\dim \mathscr{B}(\mathfrak{L}_q(1, \mathscr{G})) = p^{\mathscr{G}+3}$; indeed a PBW-basis is

$$B = \{x_1^{m_1} y_1^{m_2} z_{\mathscr{G}}^{n_{\mathscr{G}}} \dots z_1^{n_1} z_0^{n_0} : 0 \le m_i, n_j < p\}.$$

The minimal bosonization. Assume that q is a root of 1. Set $d := \operatorname{ord} q$; then $(d, p) = 1$. Fix $f \in \mathbb{Z}_{>0}\, pd$. Let

$$\Gamma = \mathbb{Z}/f \times \mathbb{Z}/f = \langle g_1 \rangle \oplus \langle g_2 \rangle, \qquad \text{where } \operatorname{ord} g_1 = \operatorname{ord} g_2 = f.$$

Let $\varGamma = \mathbb{Z}/p = \langle g \rangle$. We realize $\mathfrak{L}_q(1, \mathscr{G})$ in ${}^{\Bbbk\varGamma}_{\Bbbk\varGamma}\mathcal{YD}$ by

$$\begin{aligned} g_1 \cdot x_1 &= x_1, & g_1 \cdot y_1 &= y_1 + x_1, & g_1 \cdot x_2 &= q x_2, \\ g_2 \cdot x_1 &= q^{-1} x_1, & g_2 \cdot y_1 &= q^{-1}(y_1 + a x_1), & g_2 \cdot x_2 &= x_2, \end{aligned}$$

$$\deg x_1 = g_1, \qquad \deg y_1 = g_1, \qquad \deg x_2 = g_2.$$

The Hopf algebra $H = \mathscr{B}(\mathfrak{L}_q(1, \mathscr{G})) \# \Bbbk\varGamma$ has dimension $p^{\mathscr{G}+3} f^2$.

Let $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Let $V(\mathscr{G})$ be a $\mathfrak{sl}(2)$-module of highest weight $\mathscr{G}$, with a basis $(v_n)_{n\in\mathbb{I}_{0,\mathscr{G}}}$ such that

$$E \cdot v_n = v_{n+1}, \qquad 0 \leq n < \mathscr{G}, \qquad E \cdot v_{\mathscr{G}} = 0.$$

Let $\mathfrak{l} = V(\mathscr{G}) \rtimes \Bbbk E$, a restricted Lie subalgebra of $V(\mathscr{G}) \rtimes \mathfrak{sl}(2)$.

Lemma. [A-Natale] $\mathscr{B}(\mathfrak{L}_1(1,\mathscr{G}))\#\Bbbk\Gamma$ fits into a split abelian extension

$$\Bbbk \to K \xrightarrow{\iota} \mathscr{B}(\mathfrak{L}_1(1,\mathscr{G}))\#\Bbbk\Gamma \xrightarrow{\pi} \mathfrak{u}(\mathfrak{l}) \to \Bbbk,$$

where $K = \Bbbk\langle x_1, g_1, g_2\rangle$, ι is the inclusion and π is defined by

$$\pi(x_1) = 0, \qquad \pi(y_1) = E, \qquad \pi(x_2) = v_0,$$
$$\pi(g_1) = 1, \qquad \pi(g_2) = 1.$$

Theorem. [A-Natale]
(i) The Drinfeld double $D\left(\mathscr{B}(\mathfrak{L}_1(1,\mathscr{G}))\#\Bbbk\Gamma\right)$ has fgc.
Therefore, the Hopf algebra $\mathscr{B}(\mathfrak{L}_1(1,\mathscr{G}))\#\Bbbk\Gamma$ and the Nichols algebra $\mathscr{B}(\mathfrak{L}_1(1,\mathscr{G}))$ have fgc.

(ii) The Drinfeld double $D\left(\mathscr{B}(\mathfrak{L}_q(1,\mathscr{G}))\#\Bbbk\Gamma\right)$ has fgc.
Therefore, the Hopf algebra $\mathscr{B}(\mathfrak{L}_q(1,\mathscr{G}))\#\Bbbk\Gamma$ and the Nichols algebra $\mathscr{B}(\mathfrak{L}_q(1,\mathscr{G}))$ have fgc.

Indeed, $\mathscr{B}(\mathfrak{L}_q(1,\mathscr{G}))\#\Bbbk\Gamma$ is a cocycle deformation of $\mathscr{B}(\mathfrak{L}_1(1,\mathscr{G}))\#\Bbbk\Gamma$.