On the Hopf envelope of finite-dimensional bialgebras

Based on a joint work with C. Menini and P. Saracco ($arxiv{:}2504.05821$)

Alessandro Ardizzoni

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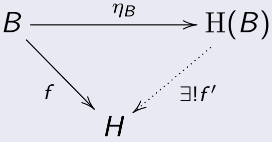
Recall that the Hopf envelope of a bialgebra B is a Hopf algebra $\mathrm{H}(B)$ with a bialgebra map $\eta_B : B \to \mathrm{H}(B)$ such that any bialgebra map $f : B \to H$ into a Hopf algebra H factors through $\mathrm{H}(B)$, i.e. there exists a unique Hopf algebra map $f' : \mathrm{H}(B) \to H$ such that $f' \circ \eta_B = f$.



Equivalently $\mathrm{H} : \mathsf{Bialg} \to \mathsf{Hopf}$ is a left adjoint of the forgetful functor.

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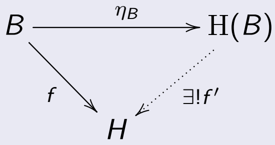

Equivalently H : Bialg $\to$ Hopf is a left adjoint of the forgetful functor.

The above notion is attributed to Manin

Mn Y.I. Manin. *Quantum groups and noncommutative geometry*, 1988.

and extends the notion of the free Hopf algebra on a coalgebra claimed by Sweedler and later proved by Takeuchi.

Sw M.E. Sweedler. *Hopf Algebras*, 1969.

Ta M. Takeuchi. *Free Hopf algebras generated by coalgebras*, 1971.

An explicit construction of $\mathrm{H}(B)$ is given in

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It is given by defining

- the family $(B_i)_{i\in\mathbb{N}}$ in Bialg: $B_0 := B$; $B_{i+1} := B_i^{\mathrm{op\,cop}}$;
- $B' = \coprod_{i\in\mathbb{N}} B_i$, the coproduct in Bialg (equiv. in Alg) of $(B_i)_{i\in\mathbb{N}}$;
- the anti-bialgebra map $S' : B' \to B'$ such that $S' \circ \iota_i = \iota_{i+1}$ where $\iota_i : B_i \to B'$ are the coproduct injections;
- $I := \langle (S' * \mathrm{Id} - u'\varepsilon')(x_i), (\mathrm{Id} * S' - u'\varepsilon')(x_i) \mid x_i \in \iota_i(B_i), i \in \mathbb{N}\rangle$;
- $\mathrm{H}(B) = \frac{B'}{I}$;
- $\eta_B : B \to \mathrm{H}(B)$ by $\eta_B(b) = \iota_0(b) + I$;
- $S : \mathrm{H}(B) \to \mathrm{H}(B)$, by $S(b' + I) = S'(b') + I$, $b' \in B'$;

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Evidently, the construction above is not so handy: for instance the coproduct $\coprod_{i\in\mathbb{N}} A_i$ in Alg of a family of algebras $(A_i)_{i\in\mathbb{N}}$ is obtained as a suitable quotient of the tensor algebra $T(\oplus_{i\in N} A_i)$.

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It is therefore interesting to look for simpler descriptions, when available.

The bialgebra map $\eta_B : B \to \mathrm{H}(B)$ is always an epimorphism in Bialg, see

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However, in general, it is not surjective as the following example shows.

Example

When M is a monoid and $B = \Bbbk M$ is the monoid algebra, it is known that $\mathrm{H}(B) = \Bbbk\mathrm{G}(M)$ where $\mathrm{G}(M)$ is the enveloping group of M while η_B is induced by the canonical map $\eta_M : M \to \mathrm{G}(M)$.
If M is abelian then $\mathrm{G}(M)$ is the Grothendieck group of M.
In particular, $\mathrm{G}(\mathbb{N}) = \mathbb{Z}$ and $\eta_{\mathbb{N}} : \mathbb{N} \to \mathbb{Z}$ is injective.

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1. to provide an alternative and simpler description of $\mathrm{H}(B)$;
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We will attach to any bialgebra B a suitable quotient bialgebra $Q(B)$ which will coincide with $\mathrm{H}(B)$ in the finite-dimensional case.

First consider $B \otimes B$ as a right B-module through the diagonal action, i.e. $(x \otimes y)b := xb_1 \otimes yb_2, \forall x, y, b \in B$, summation understood.

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$B \oslash B$ is a quotient coalgebra of $B \otimes B^{\mathrm{cop}}$ and the canonical map $i_B : B \to B \oslash B, x \mapsto x \oslash 1,$ is a coalgebra map.

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Example

When B has antipode S, then i_B is bijective with $i_B^{-1}(x \oslash y) = xS(y)$.

Definition

For an arbitrary bialgebra B we define

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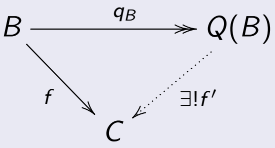
Proposition (Key property of $Q(B)$)

Any bialgebra map $f : B \to C$ into a bialgebra C with i_C injective (e.g. C is Hopf) factors through $Q(B)$ i.e. there is a unique bialgebra map $f' : Q(B) \to C$ such that $f' \circ q_B = f$.



Proof. One has $(f \oslash f) \circ i_B = i_C \circ f$ so that $\ker(i_B) \subseteq \ker(i_C \circ f) = \ker(f)$. □

In order to prove that $\mathrm{H}(B) = Q(B)$ for B f.d., we need the following.

Lemma

If B is a finite-dimensional bialgebra, there is a minimum $n \in \mathbb{N}$ and a map $S \in \mathrm{End}_{\Bbbk}(B)$ with $S * \mathrm{Id}^{*n+1} = \mathrm{Id}^{*n} = \mathrm{Id}^{*n+1} * S$.
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Proof.

Since B is f.d., then so is the convolution algebra $\mathrm{End}_{\Bbbk}(B)$.
Hence $\mathrm{Id} \in \mathrm{End}_{\Bbbk}(B)$ is algebraic over $\Bbbk$.
Thus there is a polynomial $f \in \Bbbk[X]$ such that $f(\mathrm{Id}) = 0$.
If n is the degree of the smallest power of X occurring in f, we can assume its coefficient to be 1 and rewrite $f = g \cdot X^{n+1} + X^n$ for some $g \in \Bbbk[X]$.
Then $0 = f(\mathrm{Id}) = g(\mathrm{Id}) * \mathrm{Id}^{*n+1} + \mathrm{Id}^{*n}$ and hence $S * \mathrm{Id}^{*n+1} = \mathrm{Id}^{*n}$ where we set $S := -g(\mathrm{Id}) \in \mathrm{End}_{\Bbbk}(B)$.
Since $Xg = gX$ we have $\mathrm{Id} * S = S * \mathrm{Id}$. Thus $\mathrm{Id}^{*n+1} * S = \mathrm{Id}^{*n}$. □

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Example

Fix $n \in \mathbb{N}$. Consider the monoid $G = \langle x \mid x^{n+1} = x^n \rangle$, presented by the generator x with relation $x^{n+1} = x^n$. Let $\Bbbk G$ be the monoid algebra. Then $\mathrm{Id}^{*n+1} = \mathrm{Id}^{*n}$ and $\Bbbk G$ is an n-Hopf algebra with n-antipode $S = u \circ \varepsilon$.

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Let B be a 2-Hopf algebra with 2-antipode S.
Then $S * \mathrm{Id}^{*3} = \mathrm{Id}^{*2}$ i.e. $S(y_1)y_2y_3y_4 = y_1y_2, \forall y \in B$ so that

$$xS(y) \oslash 1 = x\color{red}{S(y_1)y_2y_3y_4}\color{black} \oslash y_7y_6y_5 = xy_1y_2 \oslash y_5y_4y_3 = x \oslash y.$$

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A similar argument for an arbitrary $n \in \mathbb{N}$ leads to

Proposition

Let B be a n-Hopf algebra with n-antipode S.
Then $xS(y) \oslash 1 = x \oslash y$ for every $x, y \in B$.
In particular, the canonical map $i_B : B \to B \oslash B, x \mapsto x \oslash 1$, is surjective.

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Proof Sketch. **Step1** We have seen that B has an n-antipode S for some $n \in \mathbb{N}$ and that $xS(y) \oslash 1 = x \oslash y, \forall x, y \in B$. In particular $x_1 S(x_2) \oslash 1 = x_1 \oslash x_2 = \varepsilon(x)1 \oslash 1$ so that $x_1 S(x_1) - \varepsilon(x)1 \in \ker(i_B)$.
Since $\ker(q_B) = \langle \ker(i_B) \rangle$, we get $q_B(x_1) q_B S(x_1) = \varepsilon(x)1$.
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Step2 Since $B' := Q(B)$ is a quotient of B it is finite-dimensional too. Hence it has an n'-antipode S' for some $n' \in \mathbb{N}$. Now

$$\mathrm{Id}^{*n'} = S' * \mathrm{Id}^{*n'+1} \Rightarrow \mathrm{Id}^{*n'} q_B = (S' * \mathrm{Id}^{*n'+1}) q_B \Rightarrow q_B^{*n'} = (S' q_B) * q_B^{*n'+1}.$$

By Step1, we can cancel $q_B^{*n'}$ on the right.
Therefore $S' q_B$ and q_B are mutual convolution inverses.

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Step3 Since q_B is surjective we get that S' and Id are mutual convolution inverses so $Q(B)$ is a Hopf algebra. Therefore its key property is just the universal property of $\mathrm{H}(B)$ so that $Q(B) = \mathrm{H}(B)$. □

Two Examples

As an example of n-Hopf algebra, we cosidered $B = \Bbbk\langle x \mid x^{n+1} = x^n \rangle$.
More generally, for $p \in \mathbb{N} \setminus \{0\}$, we have the finite-dimensional bialgebra

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One proves that $\ker(i_B) = \langle x^p - 1 \rangle$ so that

$$\mathrm{H}(B) \overset{B \text{ f.d.}}{=} Q(B) = \frac{B}{\langle \ker(i_B) \rangle} = \frac{B}{\langle x^p - 1 \rangle} \cong \Bbbk\langle x \mid x^p = 1 \rangle = \Bbbk C_p$$

where C_p is the cyclic group of order p.

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In the particular case when $n = 2$ and $p = 3$, we get $B = \Bbbk\langle x \mid x^5 = x^2 \rangle$ and $S = \mathrm{Id}^{*2}$. One easily checks that $S(x^4) = S(x)$ so that S is not injective whence not even surjective.

The second example we want to illustrate is the f.d. bialgebra

$$B = \Bbbk \langle x, y \mid yx = -xy, x^3 = x, y^2 = 0 \rangle$$

which is 6-dimensional with basis $\{1, x, x^2, y, xy, x^2 y\}$.
Its coalgebra structure is uniquely determined by

$$\Delta(x) = x \otimes x \quad \text{and} \quad \Delta(y) = x \otimes y + y \otimes 1.$$

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which is 6-dimensional with basis $\left\{1, x, x^2, y, xy, x^2 y\right\}$.
Its coalgebra structure is uniquely determined by

$$\Delta(x) = x \otimes x \quad \text{and} \quad \Delta(y) = x \otimes y + y \otimes 1.$$

One proves that $\ker(i_B) = \langle x^2 - 1 \rangle$ so that

$$\mathrm{H}(B) \overset{B \text{ f.d.}}{=} Q(B) = \frac{B}{\langle \ker(i_B) \rangle} = \frac{B}{\langle x^2 - 1 \rangle} \cong H_4$$

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Moreover B has a 1-antipode $S : B \to B$ which is an anti-algebra map with $S^4 = \mathrm{Id}$. It is defined by $S(x) = x$ and $S(y) = \left(1 - x - x^2\right) y$.

Beyond the finite-dimensional case

Under appropriate assumptions, our construction extends to the infinite-dimensional case.

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For instance we have the following

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As a consequence, since B is left Artinian so is its quotient $\mathrm{H}(B) = Q(B)$.
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Remark

The proof of the above theorem does not use the notion of n-Hopf algebra since we don't know whether left Artinian implies n-Hopf.

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Example

Let A be a $\Bbbk$-algebra and consider the product algebra $B := A \times \Bbbk$.
Note that if A is left Artinian and infinite-dimensional (e.g $A = \mathbb{R}$, $\Bbbk = \mathbb{Q}$) then so is B (left ideals in B are $I \times 0$ and $I \times \Bbbk$ where I is a left ideal in A).
Then B becomes a bialgebra by setting, $\forall a \in A, k \in \Bbbk$,

$$\Delta(a,k) = (1,1) \otimes (a,0) + (a,k) \otimes (0,1) \qquad \text{and} \qquad \varepsilon(a,k) = k.$$

One gets that $S := u_B \circ \varepsilon_B$ is a 1-antipode and $\ker(i_B) = \ker(\varepsilon_B)$. Hence

$$\mathrm{H}(B) \overset{B \text{ left.Art.}}{=} Q(B) = \frac{B}{\langle \ker(i_B) \rangle} = \frac{B}{\ker(\varepsilon_B)} \cong \Bbbk$$

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The previous example applies also to A finite-dimensional...

If $A = \Bbbk\langle c \mid c^n = 1\rangle$, $n > 1$, we get the bialgebra
$B = A \times \Bbbk = \Bbbk\langle x \mid x^{n+1} = x\rangle$ where $x := (c, 0)$,
$\Delta(x) = 1 \otimes x + x \otimes (1 - x^n)$ and $\varepsilon(x) = 0$.

We noticed that $\eta_B : B \to \mathrm{H}(B)$ is not surjective in general.
As a consequence $\mathrm{H}(B)$ and $Q(B)$ do not always coincide.
EXAMPLE: $\mathrm{H}(\Bbbk\mathbb{N}) = \Bbbk\mathbb{Z} \neq \Bbbk\mathbb{N} = Q(\Bbbk\mathbb{N})$.

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Still we can iteratively reconstruct $\mathrm{H}(B)$ as follows.

For an arbitrary bialgebra B, define iteratively $Q^{n+1}(B) = Q(Q^n(B))$.
This gives a sequence of bialgebras

$$B \xrightarrow{q_B} Q(B) \xrightarrow{q_{Q(B)}} Q^2(B) \xrightarrow{q_{Q^2(B)}} Q^3(B) \xrightarrow{q_{Q^3(B)}} \cdots .$$

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We can prove that

Theorem

Let B be an n-Hopf algebra. Then $\mathrm{H}(B) = Q^\infty(B)$.

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Proposition

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For $B = \Bbbk M$ a monoid algebra, i_B is injective iff M is right cancellative.

Remark

Let $B = \Bbbk M$ be a monoid algebra.
Since $Q^{\infty}(B)$ is a quotient of B we get $Q^{\infty}(B) = \Bbbk C$ for some monoid C.
By the previous result, $i_{Q^{\infty}(B)}$ injective means that C is right cancellative.
The key property of $Q(B)$ now implies that C is the maximal right cancellative monoid homomorphic image of M.

Thus $Q^{\infty}(B)$ can be regarded as a bialgebra counterpart of the maximal right cancellative monoid homomorphic image.

THANK YOU!