Orbit Categories and Hecke algebras for Hopf algebras

Hopf 25 (Brussels 25th April 2025)

Andrew Baker (University of Glasgow)

Printable version available at
`https://www.maths.gla.ac.uk/∼ajb/dvi-ps/Talks/Hopf25.pdf`

last updated 25/04/2025

Outline of talk

This talk is a preliminary report on work in progress. Comments welcomed.

- Orbit categories and Hecke algebras for groups.
- Orbit categories and their linearisations for (pointed) Hopf algebras.
- Hecke algebras for (finite dimensional) Hopf algebra pairs.
- Things to do: Mackey & Green functors, cohomological aspects,. . . .

Recollections on orbit categories for groups

Let G be a (finite) group and $\mathfrak{F}$ a family of subgroups which is closed under conjugation (maybe with additional properties). The *orbit category* $\mathcal{O}_{G,\mathfrak{F}}$ has as its objects the sets of left cosets G/K ($K \in \mathfrak{F}$), and morphisms the G-equivariant maps $f\colon G/L \to G/K$ ($K, L \in \mathfrak{F}$). Here f is determined by its value on the coset $\overline{1} = 1L$, $f(\overline{1}) = \overline{x} = xK$ say, where for $\ell \in L$, $\ell\overline{x} = \overline{x}$; hence $x^{-1}Lx \leqslant K$ or equivalently $L \leqslant xKx^{-1}$.

We can linearise $\mathcal{O}_{G,\mathfrak{F}}$ over a field $\Bbbk$ using the functors

$$\Bbbk(-)\colon \mathcal{O}_{G,\mathfrak{F}} \to \mathbf{Mod}_{\Bbbk G} \to \mathbf{Vect}_{\Bbbk}$$

which send G/K to the permutation $\Bbbk G$-module $\Bbbk G/K$ and morphisms to the induced $\Bbbk G$-module or $\Bbbk$-linear maps. Now $\Bbbk G$ is a Hopf algebra and for any subgroup $K \leqslant G$, the quotient map $\Bbbk G \to \Bbbk G/K$ is a homomorphism of $\Bbbk G$-module coalgebras. If $K, L \in \mathfrak{F}$ there is a natural bijection

$$\mathbf{Coalg}_{\Bbbk}^{\Bbbk G}(\Bbbk G/L, \Bbbk G/K) \longleftrightarrow \mathcal{O}_{G,\mathfrak{F}}(G/L, G/K).$$

The classical notion of Hecke algebra is the vector space $\Bbbk H\backslash G/H$ spanned by the double cosets of H equipped with a multiplication making it isomorphic to the opposite algebra $\mathrm{End}_{\Bbbk G}(\Bbbk G/H)^{\mathrm{op}}$. Here an endomorphism $f \in \mathrm{End}_{\Bbbk G}(\Bbbk G/H)$ is determined by its value on $\overline{1}$ and this lies in the H-invariant subspace ${}^{H}(\Bbbk G/H)$. If G is finite every H-invariant is a linear combination of H-orbit sums

$$\sum_{h\colon\, H/H\cap xHx^{-1}} hxH$$

taken over a complete set of representatives of $H/H \cap xHx^{-1}$, and this is identified with the double coset HxH.

The multiplication: For $f_x \in \mathrm{End}_{\Bbbk G}(\Bbbk G/H)$ with $f_x(\bar{1}) = HxH$,

$$f_x \circ f_y(\bar{1}) = \sum_{k \colon H/H \cap yHy^{-1}} kyf_x(\bar{1}) = \sum_{\substack{x \colon H/H \cap xHx^{-1} \\ k \colon H/H \cap yHy^{-1}}} kyhx\bar{1}.$$

So identifying HxH with f_x we obtain a formula involving the sum over all double cosets

$$HyH \cdot HxH = \sum_z c^z_{y,x} HzH,$$

where $c^z_{y,x}$ is a certain numerical function of x, y, z.

Recollections on Hopf algebras and quotient coalgebras

For a $\Bbbk$-coalgebra C let C_0 be its coradical, i.e., the sum of all its simple subcoalgebras. For a surjective coalgebra homomorphism $\theta\colon C \to D$, $D_0 \subseteq \theta C_0$. When C is pointed this implies that D_0 is spanned by group-like elements so D is also pointed.

For a Hopf algebra H and subHopf algebra $K \subseteq H$,

$$H/\!/K = H/HK^+ \cong H \otimes_K \Bbbk$$

is a quotient H-module coalgebra where $K^+ = \ker \varepsilon$ (the kernel of the counit). If H is pointed so is $H/\!/K$ and its group-like elements are images of group-likes of H under the quotient map. An H-module coalgebra homomorphism $\alpha\colon H/\!/L \to H/\!/K$ is determined by the group-like element $\alpha(\overline{1})$ where $\overline{1} = 1 + HL^+$, so $\alpha(\overline{1}) = \overline{a} = a + HK^+$ for some $a \in H$; if a is group-like then it is also a unit. Also, if $\ell \in L^+$ then $\overline{\ell a} = \overline{0}$; so if a is invertible, $a^{-1}La \subseteq HK^+$ or equivalently,

$$L \subseteq H(aK^+a^{-1}) = H(aKa^{-1})^+.$$

Orbit categories for Hopf algebras

Now let H be a finite dimensional (pointed) Hopf algebra over the field $\Bbbk$ and let $\mathfrak{G}$ be a family of subHopf algebras closed under adjoint actions (maybe with other conditions). The *orbit category* $\mathcal{O}_{H,\mathfrak{G}}$ has objects the $H/\!/K$ ($K \in \mathfrak{G}$) and morphisms given by

$$\mathcal{O}_{H,\mathfrak{G}}(H/\!/L, H/\!/K) = \mathbf{Coalg}_{\Bbbk}^{H}(H/\!/L, H/\!/K),$$

the set of H-module coalgebra homomorphisms $H/\!/L \to H/\!/K$. An element $\alpha \in \mathbf{Coalg}_{\Bbbk}^{H}(H/\!/L, H/\!/K)$ is determined by $\alpha(\overline{1})$ which is an L-invariant (i.e., it is annihilated by L^{+}) group-like element of $H/\!/K$. We will denote the set of such elements by $\mathrm{Gp}^{L}(H/\!/K)$ and identify it with $\mathbf{Coalg}_{\Bbbk}^{H}(H/\!/L, H/\!/K)$.

There is an inclusion

$$\mathbf{Coalg}^H_{\mathbb{k}}(H/\!/L, H/\!/K) = \mathrm{Gp}^L(H/\!/K) \hookrightarrow \mathbf{Mod}_L(\mathbb{k}, H/\!/K)$$

where we send $\overline{x} \in \mathrm{Gp}^L(H/\!/K)$ to the linear map $\mathbb{k} \to H/\!/K$ sending 1 to $\overline{x}$. This is an L-module homomorphism and

$$\mathbf{Mod}_L(\mathbb{k}, H/\!/K) \cong \mathbf{Mod}_H(H \otimes_L \mathbb{k}, H/\!/K) = \mathbf{Mod}_H(H/\!/L, H/\!/K),$$

so we obtain a map

$$\mathbf{Coalg}^H_{\mathbb{k}}(H/\!/L, H/\!/K) \hookrightarrow \mathbf{Mod}_H(H/\!/L, H/\!/K).$$

Consider an H-module coalgebra homomorphism

$$H/\!/L \to H/\!/K; \quad \overline{h} \mapsto \overline{hx}$$

where $L^+x \in HK^+$ and $x \in H$ with $\overline{x}$ group-like.
If H is pointed then we can take x to be group-like and so invertible. Since $\overline{h} = \overline{(hx^{-1})x}$, this homomorphism is surjective. If $\dim_{\mathbb{k}} L = \dim_{\mathbb{k}} K$, such a homomorphism is an isomorphism.

Hopf Hecke algebras

Definition: The *Hopf Hecke algebra* of a pair $K \subseteq H$ of finite dimensional Hopf algebras is the algebra

$$\mathcal{H}(H,K) = \mathrm{End}_H(H/\!/K)^{\mathrm{op}}.$$

An element $\alpha \in \mathcal{H}(H,K)$ is an H-module homomorphism with $\alpha(\bar{1}) \in {}^K(H/\!/K)$, so $\alpha(\bar{1}) = \bar{a}$ where $K^+a \subseteq HK^+$. For another element β with $\beta(\bar{1}) = \bar{b}$, $\beta \circ \alpha(\bar{1}) = \overline{ab}$. This gives an algebra structure on ${}^K(H/\!/K)$ making it isomorphic to $\mathrm{End}_H(H/\!/K)^{\mathrm{op}}$. Alternatively, ${}^K(H/\!/K)$ can be viewed as the quotient algebra of the idealizer of the left ideal HK^+ in H.

To get results analogous to known ones for Hecke algebras of finite groups some assumptions are required on the pair $K \subseteq H$ of finite dimensional Hopf algebras

The pair (H,K) is a *finite Frobenius Hopf pair* if H is unimodular and involutory (hence K is also involutory); it follows that $K \subseteq H$ is a Frobenius extension.
As $\dim {}^H(H/\!/K) = 1$, we can choose $t_0 \in H$ projecting to a non-zero element $\overline{t_0} \in {}^H(H/\!/K)$. Such a pair (H,K) is *regular* if

- the *relative trace* $\tau_K^H = t_0 \cdot \colon {}^K H \to {}^H H = \int_H^{\mathrm{l}}$ is non-trivial on ${}^H H \subseteq {}^K H$, i.e., $\varepsilon(t_0) \neq 0$.

In that case we can assume $\varepsilon(t_0) = 1$ and $\overline{t_0^2} = \overline{t_0} \in \mathcal{H}(H,K)$, and it can be shown that $\overline{t_0}$ is central (this uses the involution to be discussed later).
For a left H-module M, $\tau_K^H \colon {}^K M \to {}^H M$ restricts to the identity function on ${}^H M \subseteq {}^K M$. So a short exact sequence of H-modules

$$0 \to L \to M \to N \to 0$$

splits iff the short exact sequence of K-modules

$$0 \to L\downarrow_K^H \to M\downarrow_K^H \to N\downarrow_K^H \to 0$$

splits.

These conditions imply there is an isomorphism of left H-modules

$$H/\!/K \xrightarrow{\cong} (H/\!/K)^*$$

which we make explicit as follows.
Let $0 \neq s_0 \in \int_K^{\mathrm{l}}$. Then there is an isomorphism of left H-modules

$$H/\!/K \xrightarrow{\cong} Hs_0; \quad \overline{h} \longleftrightarrow hs_0.$$

It can be shown that $t_0 s_0 \in \int_H^{\mathrm{l}}$ and $t_0 s_0 = \pm\chi(s_0)t_0$ where $\chi(s_0) \in \int_K^{\mathrm{r}}$.
For a Frobenius form λ on H define an left H-module isomorphism

$$H/\!/K \xrightarrow{\cong} (H/\!/K)^*; \quad \overline{h} \longleftrightarrow \lambda((-)hs_0).$$

An algebra *involution* will mean a self-inverse anti-algebra homomorphism.

Theorem

There is an algebra involution

$$\widehat{(\ )}\colon \mathcal{H}(H,K) \to \mathcal{H}(H,K); \quad \widehat{\overline{a}} = \overline{\widehat{a}}$$

characterised by $\widehat{a}s_0 = \chi(s_0)\chi(a)$*, where* $\widehat{a} \in H$ *is only well defined modulo* HK^+*.*

For the case of a classical Hecke algebra for a pair of finite groups $H \leqslant G$, this involution of $\Bbbk H\backslash G/H$ is given by $HgH \mapsto Hg^{-1}H$. There is a *trace form* Λ on $\mathcal{H}(H,K) \cong {}^{K}(H/\!/K) \subseteq H/\!/K$, given by $\overline{h} \mapsto \lambda(hs_0)$ and Λ has an associated non-degenerate pairing on $\mathcal{H}(H,K)$ given by

$$(\overline{a},\overline{b}) \mapsto \lambda(a\widehat{b}s_0) = \lambda(a\chi(s_0)\chi(a)).$$

If K is also unimodular then $\chi(s_0) = \pm s_0$.

The following commutative diagram defines $\widehat{(\ )}$.

$$
\begin{array}{ccc}
\mathcal{H}(H,K) & \xrightarrow{\widehat{(\ )}} & \mathcal{H}(H,K) \\
\| & & \| \\
{}^{K}(H/\!/K) & & {}^{K}(H/\!/K) \\
\cong \updownarrow & & \updownarrow \cong \\
\mathrm{Hom}_K(\Bbbk, H/\!/K) & & \mathrm{Hom}_K(\Bbbk, H/\!/K) \\
\cong \updownarrow & & \updownarrow \cong \\
\mathrm{Hom}_K(\Bbbk, (H/\!/K)^*) & & \mathrm{Hom}_K(\Bbbk, (H/\!/K)^*) \\
\cong \updownarrow \text{adjunction} & & \text{adjunction} \updownarrow \cong \\
\mathrm{Hom}_{\Bbbk}(\Bbbk \otimes_K H/\!/K, \Bbbk) & & \mathrm{Hom}_{\Bbbk}(\Bbbk \otimes_K H/\!/K, \Bbbk) \\
\| & & \| \\
\mathrm{Hom}_{\Bbbk}(\Bbbk \otimes_K H \otimes_K \Bbbk, \Bbbk) & \underset{\cong}{\overset{(1\otimes\chi\otimes 1)^*}{\longleftrightarrow}} & \mathrm{Hom}_{\Bbbk}(\Bbbk \otimes_K H \otimes_K \Bbbk, \Bbbk)
\end{array}
$$

More to do: In general $\mathcal{H}(H, K)$ is not semi-simple. In the finite group case this can be characterised using conditions on the characteristic of $\mathbb{k}$ not dividing orbit lengths. This will involve better understanding of the trace form. Regularity amounts to a kind of relative semi-simplicity condition.

Mackey functors and all that

An important motivation for introducing Hecke algebras is that they act on certain types of functors on orbit categories. For example, the cohomology functor $\mathrm{Ext}^{\bullet}_{H}(\Bbbk, H/\!/K)$ has an action by $\mathcal{H}(H, K)$ and this generalises to actions on Mackey functors for an orbit category; formulating this precisely requires notions such as *rings with many objects* built out of 'double coset' constructions $\mathcal{H}(H, K, L)$ for pairs of subHopf algebras K, L of H. A Mackey functor encodes restriction along morphisms $H/\!/L \to H/\!/K$ as well as induction/transfer maps which arise from commutative diagrams of the form

$$\begin{array}{ccc} H/\!/K & \xrightarrow{f_\dagger} & H/\!/L \\ \cong\updownarrow & & \updownarrow\cong \\ (H/\!/K)^* & \xrightarrow{f^*} & (H/\!/L)^* \end{array}$$

where $f\colon H/\!/L \to H/\!/K$ is an H-module homomorphism.

In fact we have already defined a transfer/induction map for two left H-modules M and N, namely

$$\tau_K^H\colon \operatorname{Hom}_K(M,N) \to \operatorname{Hom}_H(M,N)$$

since there are identifications

$$\operatorname{Hom}_K(M,N) = {}^K\operatorname{Hom}_{\Bbbk}(M,N), \quad \operatorname{Hom}_H(M,N) = {}^H\operatorname{Hom}_{\Bbbk}(M,N).$$

Here the map is given in Sweedler notation by

$$\tau_K^H(f) = t_0 \cdot f = \sum (t_0)_{(1)} f(\chi((t_0)_{(2)})(-)).$$

If $M = \Bbbk$ this induces a transfer map $\operatorname{Ext}_K^{\bullet}(\Bbbk, N) \to \operatorname{Ext}_H^{\bullet}(\Bbbk, N)$.

Some references

- A. Krieg, Hecke Algebras, Mem. Amer. Math. Soc., **87** no. 435 (1990).
- M. Lorenz, A Tour of Representation Theory, Grad. Studies in Math. **193**, Amer. Math. Soc. (2018).
- B. Mitchell, Rings with several objects, Adv. in Math. **8** (1972), 1–161.

Thank you for listening!