Twists of reflection groups and Cherednik algebras

Hopf25, ULB, Brussels, 24.4.25

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`arXiv:2501.06673` by E. Jones-Healey, YB + work in progress by YB

MANCHESTER 1824

DKO Travel Fund

Reflection groups after Chevalley, Shephard, Todd, Serre

Theorem[1]. Let $\dim_{\mathbb{C}} V = n$. For a finite group $G \subset GL(V)$, TFAE:

subalgebra   subcoalgebra

- $S(V)^G$ is a polynomial algebra.
- $S(V) \cong S(V)^G \otimes \mathrm{Har}_G$ i.e. $S(V)$ is free over the invariants.
- G is generated by (complex) reflections.

$s : V \to V$
$\dim (s - \mathrm{Id})V = 1$

Complex reflection groups of rank n are classified according to Shephard-Todd: infinite family $G(m, p, n)$ of imprimitive groups, and several exceptional cases. (*This includes all Weyl and Coxeter groups*)

[1] C. Chevalley. "Invariants of finite groups generated by reflections". In: *Amer. J. Math.* 77 (1955).

The Shephard-Todd groups $G(m, p, n)$

$G(m, p, n)$, where $p|m$, is the group of $n \times n$ matrices with

- $n - 1$ zeros and one mth root of unity in every row;
- ditto in every column;
- $\prod$(all non-zero entries) = an (m/p)th root of unity.

$G(1, 1, n) \cong S_n$ is the Coxeter group of type A_{n-1};

Hyperoctahedral group

$G(2, 1, n) \cong \mathbb{Z}_2^n \rtimes S_n$ is the Coxeter group of type B_n;

$G(2, 2, n) \subset G(2, 1, n)$ is the Coxeter group of type D_n;

$G(m, 1, n)$ is the wreath product group $\mathbb{Z}_m^n \rtimes S_n$.

Drinfeld's problem

Drinfeld[2] posed the following problem: let finite G act on V and let $\kappa : V \times V \to \mathbb{C}G$ be a skew-symmetric bilinear form.

Consider the relations $R = \{xy - yx - \kappa(x,y) : x, y \in V\}$.

Call $A_\kappa(G,V) = T(V) \rtimes \mathbb{C}G/(R)$ a degenerate affine Hecke algebra if it factorizes as $\underline{S(V) \cdot \mathbb{C}G}$.

(Clearly, $A_0(G,V) = S(V) \rtimes \mathbb{C}G$)

What are degenerate affine Hecke algebras? Too difficult!

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[2] V. G. Drinfeld. "Degenerate affine Hecke algebras and Yangians". In: *Funktsional. Anal. i Prilozhen.* 20.1 (1986).

Solution: rational Cherednik algebra

Etingof and Ginzburg[3] solved Drinfeld's problem assuming that V has a G-invariant nondegenerate symplectic form.

E.g., $V \oplus V^*$ has canonical such form, invariant w.r.t. $G \subset GL(V) \rightsquigarrow$ **rational Cherednik algebra** $H_\kappa(G, V) = S(V) \cdot \mathbb{C}G \cdot S(V^*)$.

Essentially G must be a reflection group, and $\kappa(y, x) = t\langle y, x\rangle + \sum_s c_s \langle y, (1 - s)x\rangle s$ where s runs over the reflections in G.

(reflection groups after Etingof-Ginzburg)

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[3] P. Etingof and V. Ginzburg. "Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism". In: *Invent. Math.* 147.2 (2002).

Attempts to construct a q-analog of reflection groups

Two such simultaneous and independent attempts.

Kirkman, Kuzmanovich and Zhang: replace "$S(V)^G$ is polynomial" in the Chev-Shep-Todd by: G acts on $S_q(V) = \mathbb{C}\langle x_1, \dots, x_n | x_j x_i = q_{ij} x_i x_j \forall i < j \rangle$ so that $\text{gl.dim}\, A^G < \infty$.

Theorem[4]: such G are generated by **mystic reflections**.

The basic new case is $q_{ij} = -1 \; \forall i < j$.

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4 E. Kirkman, J. Kuzmanovich, and J. J. Zhang. "Shephard-Todd-Chevalley Theorem for Skew Polynomial Rings". In: *Algebras and Representation Theory* 13 (2008).

Attempts to construct a q-analog of reflection groups

Berenstein, B.: replace $S(V)$, $S(V^*)$ in rational Cherednik algebra by $S_q(V)$, $S_q(V^*)$.

Theorem[5]: these q-Cherednik algebras are bosonizations of **braided Cherednik algebras**, with G explicitly listed.

The basic case of a braided Cherednik algebra is $q_{ij} = -1\ \forall i < j$.

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5 Y. Bazlov and A. Berenstein. "Noncommutative Dunkl operators and braided Cherednik algebras". In: *Selecta Mathematica* 14 (2009).

Both attempts give the same groups

Theorem[6]: the braided Cherednik algebras at $q = -1$ exist over $\mu G(m, p, n)$ where m is even, $p|m$: the group of $n \times n$ matrices g with

- $n - 1$ zeros and one mth root of unity in every row;
- ditto in every column;
- $\det g$ = an (m/p)th root of unity.

The nicest example in rank n is $\mu G(2, 2, n)$ = group of elements of determinant 1 (even elements) in the Coxeter group of type B_n.

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6 Y. Bazlov and A. Berenstein. "Mystic reflection groups". English. In: *Symmetry, Integrability and Geometry: Methods and Applications* 10 (2014).

More on $\mu G(2,2,n)$

This room is named after[7]

$G = \mu G(2,2,n)$ is generated by $\sigma_1, \dots, \sigma_{n-1}$ subject to:

- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$; σ_i, σ_j commute if $j > i + 1$;
- $\sigma_i^4 = 1$;
- $\sigma_1^2, \dots, \sigma_{n-1}^2$ pairwise commute.

σ_i acts on $\mathbb{C}_{-1}[x_1, \dots, x_n]$ via $x_i \mapsto -x_{i+1}$, $x_{i+1} \mapsto x_i$, $x_k \mapsto x_k$, $k \neq i, i+1$.
Same for $y_1, \dots, y_n$. + Commutation rel$^{\text{ns}}$:

$\underline{H}_{t,c}(G) = \mathbb{C}_{-1}[x_1, \dots, x_n] \cdot \mathbb{C}G \cdot \mathbb{C}_{-1}[y_1, \dots, y_n]$

[7] J. Tits “Normalisateurs de Tores I. Groupes de Coxeter Étendus”. In: *Journal of Algebra* 4.1 (1966).

Minimal example: $\mu G(2,2,2)$

The group $G = \mu G(2,2,2)$ is $\langle \sigma | \sigma^4 = 1 \rangle$.

Braided Cherednik algebra $\underline{H}_{t,c} = \langle x_1, x_2, \sigma, y_1, y_2 \rangle /$

- $x_1 x_2 = -x_2 x_1$, $y_1 y_2 = -y_2 y_1$, $\sigma^4 = 1$
- $\sigma x_1 = x_2 \sigma$, $\sigma x_2 = -x_1 \sigma$, same for y
- $y_1 x_2 + x_2 y_1 = c\sigma$, $y_2 x_1 + x_1 y_2 = c\sigma^{-1}$
- $y_1 x_1 - x_1 y_1 = t + c\sigma$, $y_2 x_2 - x_2 y_2 = t + c\sigma^{-1}$.

Note that this algebra is defined over $\mathbb{Q}$.

Twisting the product in an algebra

A is a module algebra for a Hopf algebra H. Suppose $J \in H^{\otimes 2}$ is a **twist,** that is J is invertible and

- $J_{12} \cdot (\Delta \otimes \mathrm{id})J = J_{23} \cdot (\mathrm{id} \otimes \Delta)J$
- $(\epsilon \otimes \mathrm{id})J = 1 = (\mathrm{id} \otimes \epsilon)J$

(Example: a quasitriangular structure on a sub-Hopf algebra.)
Then H^J with $\Delta_J x = J(\Delta x)J^{-1}$ is a Hopf algebra.

If $J^{-1} = j' \otimes j''$ (summation understood) then:

A_J = vector space A with product $a * b = j'(a)j''(b)$ is an H^J-module algebra.

Properties of A preserved by twist

NB: properties of H which persist in H were studied e.g. by [AljEtiGelNik'02] - not discussed today.

[Mont'05]: "A is fin. generated"; if H is f.d. "A is Noetherian", "A is PI"; [GuilKassMasu'12] H is f.d. unimodular: "A is semisimple"; H acts linearly: "A is a quadratic algebra"

Jones-Healey[8]:"A is Koszul"

The Koszul property is closely related to **deformations** of quadratic algebras [Drinfeld, Braverman-Gaitsgory, Ginzburg, Etingof etc]

[8] Edward Jones-Healey. "Drinfeld twists of Koszul algebras". In: *Comm. Algebra* 52.8 (2024).

A twist can preserve a factorization

An **algebra factorization** is subalgebras B, C of A such that the composition $B \otimes C \hookrightarrow A \otimes A \xrightarrow{m_A} A$ is an isomorphism of vector spaces (*earlier written as* $B \cdot C$).

Lemma: if B, C are H-submodule algebras then $A_J = B_J \cdot C_J$.

Lemma: if $A = A^H \cdot V$ where V is an H-subspace, then $A_J = (A_J)^H \cdot V$, and in addition $(A_J)^H \cong A^H$ as algebras.

Corollary: if A is free over A^H then so is A_J.

Therefore, if G is a reflection group and J is a twist on $\Bbbk G$ then $S(V)_J$ is free over $S(V)^{\Bbbk G^J}$.

Skew poly's and braided Cherednik

$A = \Bbbk[x_1, \dots, x_n]$ is $\Bbbk\mathbb{Z}_2^n = \langle t_1, \dots, t_n \,|\, t_i^2 = 1 \rangle$ -module algebra

$J = \prod_{1 \le j < i \le n} \frac{1}{2}(1 \otimes 1 + t_i \otimes 1 + 1 \otimes t_j - t_i \otimes t_j)$ is a twist on $\Bbbk\mathbb{Z}_2^n$

$A_J = \Bbbk_{-1}[x_1, \dots, x_n]$ — that is, the twist of commutative polynomials gives skew polynomials.

Question: can we twist, by J, the rational Cherednik algebra $H_{t,c}(G) = \Bbbk[x_1, \dots, x_n] \cdot \Bbbk G \cdot \Bbbk[y_1, \dots, y_n]$? Does $\Bbbk\mathbb{Z}_2^n$ act on $\Bbbk G$?

Skew poly's and braided Cherednik 2

Theorem[9]: assume m even $\Bbbk \supset \mathbb{Q}(\sqrt[m]{1})$, $G = G(m,p,n)$ with $p|m$.

We have $\mathbb{Z}_2^n \subset G(m,1,n) \trianglerighteq G(m,p,n)$ so $\mathbb{Z}_2^n$ acts on $G(m,p,n)$ by conjugation, and this action extends to $H_{t,c}(G)$.

And $H_{t,c}(G)_J \cong \underline{H}_{t,-c}(\mu G)$ via $\begin{cases} x_i \mapsto x_i \\ y_i \mapsto y_i \\ s_i \mapsto \sigma_i \end{cases}$

$s_i = (x_i \leftrightarrow x_{i+1})$ transposition

9 Y. Bazlov et al. "Twists of rational Cherednik algebras". In: *The Quarterly Journal of Mathematics* 74.2 (2023).

What happens with the group algebras?

① $H = \mathbb{k}G(m,1,n)$ is the twisting Hopf algebra.

$$\mathbb{k}G(m,1,n) = \mathbb{k}(\mathbb{Z}_m^n \rtimes S_n).$$

What is H^J?

Answer: $\mathbb{k}_\mu G(m,1,n)$

② $\mathcal{H}_{t,c} = \mathbb{k}[x_1,\dots,x_n] \cdot \mathbb{k}G(m,p,n) \cdot \mathbb{k}[y_1,\dots,y_n]$

What is $\mathbb{k}G(m,p,n)_J$?

Answer: $\mathbb{k}_\mu G(m,p,n)$

[may not be isomorphic to $\mathbb{k}G(m,p,n)$] if $\mathbb{k} \neq \mathbb{C}$

[Interesting case: $m=2$, $p=2$]

The case of even $\frac{m}{p}$

If $\frac{m}{p}$ is even then the group $G(m,p,n)$ contains the diagonal matrices with ± 1 on the diagonal.

$$\Bbbk \mathbb{Z}_2^n \subset \Bbbk G(m,p,n) \subset \mathcal{H}_{t,c}(G(m,p,n))$$

We are in the situation of a **strongly inner action:**

DEF An action $\triangleright$ of H on A is **strongly inner** if $\exists$ algebra hom^m $u: H \to A$ such that

$$h \triangleright a = u(h_1)\, a\, u(Sh_2)$$

THM [Kulish, Mudrov '11]: for a strongly inner action,

$A_J \cong A$ via $a \mapsto (j' \triangleright a)\, u(j'')$

What happens with category $\mathcal{O}$ for $\mathcal{H}_{t,c}(G)$ under the twist?

If $\frac{m}{p}$ is even, $\mathcal{H}_{t,c}(G) \cong \mathcal{H}_{t,-c}(\mu G)$

rational Cherednik   braided Cherednik

and categories $\mathcal{O}$ for these algebras are isomorphic.

In fact, the groups G and μG are isomorphic.

$m=2, n=1$: $\mathrm{Irrep}(G(2,1,n)) \cong \{ ($  ,  $) \}$

ordered pairs of partitions
total no. of boxes is n

What happens with category $\mathcal{O}$ for $\mathcal{H}_{t,c}(G)$ under the twist? (contd.)

$m=2,\ n=1$

So we have two isomorphic categories, and isomorphism classes of simple objects in each category are labeled by $\left(\ydiagram{1,1}\,,\ \ydiagram{3,2}\right)$.

What does the isomorphism / automorphism do?

flip

Answer

$$X_{(\mu,\nu)} \longleftrightarrow \underline{X}_{(\mu,\nu^*)}$$

e.g. $$X_{\left(\ydiagram{1,1}\,,\ \ydiagram{3,2}\right)} \longleftrightarrow \underline{X}_{\left(\ydiagram{1,1}\,,\ \ydiagram{2,2,1}\right)}$$

What happens with finite-dimensional quotients ("baby Cherednik algebras")

$$\overline{H}_{0,c}(G) = H_{0,c}(G) \Big/ \left(\begin{array}{l} \text{invariants without constant term} \\ \text{in } \mathbb{K}[x_1, \dots, x_n], \\ \mathbb{K}[y_1, \dots, y_n] \end{array} \right)$$

[Gordon]

Answer: over $\mathbb{Q}$, twisted $\overline{H}_{0,c}(G(2,2,2))$ is not isomorphic to the untwisted one.

More general twists

Example $\quad G(3,1,n) \cong \mathbb{Z}_3^n \rtimes S_n \qquad n \geq 2$

The abelian group $\mathbb{Z}_3^n$ has the twist

$$J = \frac{1}{3} \prod_{1 \leq j < i \leq n} \left(\sum_{a,b=0}^{2} \omega^{ab} t_i^a \otimes t_j^b \right)$$

where $\omega^3 = 1$ in $\mathbb{k}$. Note that

$\mathbb{k}G(3,1,n)^J$ is not cocommutative

$\underset{\text{apparently}}{\rightsquigarrow}$ $\underline{\mathcal{H}_{t,c}} = \mathbb{k}_\omega[x_1, \dots, x_n] \otimes \underset{\text{Hopf}}{H} \otimes \mathbb{k}_\omega[y_1, \dots, y_n]$

THANK YOU.