Categories graded by group homomorphisms

HOPF25

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Introduction

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Theorem (Sözer-Virelizier 2023)

*Fusion **categories graded by crossed modules** can be used to construct 3-dimensional Homotopy Quantum Field Theories with base space a homotopy 2-type.*

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Fusion ***categories graded by crossed modules*** *can be used to construct 3-dimensional Homotopy Quantum Field Theories with base space a homotopy 2-type.*

Crossed module

A group homomorphism ‘with extra structure’.

Outline

$(\tau\colon H \to G)$

τ-graded categories

τ-graded categories with $H = 1$ $(\tau\colon H \to G)$

G-graded category

A category with subcategories $(\mathcal{C}_g)_{g\in G}$ such that

1. objects are finite direct sums $\bigoplus_{g\in G} X_g$, $X_g \in \mathcal{C}_g$.
2. $\mathrm{Hom}(X, Y) = 0$ for homogeneous X, Y unless $|X| = |Y|$.

Example: Vect_G is G-graded.

- Every G-graded vector space is of the form $V = \bigoplus_{g\in G} V_g$.
- Morphisms are by definition degree-preserving linear maps.

τ-graded categories with $G = 1$ $(\tau \colon H \to G)$

H-Hom-graded category

A category enriched in Vect_H. That is, a $\Bbbk$-linear category with

- $\mathrm{Hom}(X,Y) = \bigoplus_{h\in H} \mathrm{Hom}^h(X,Y) \in \mathrm{Vect}_H$.
- $|f' \circ f| = |f'||f|$.

Example: $\mathrm{Vect}_H^\bullet$ is H-Hom-graded.

- 1-subcategory is Vect_H *(objects, degree 1 morphisms)*.
- Degree h morphisms are linear $f \colon V \to W$ with each $f(V_a) \subseteq W_{ha}$.

τ-graded categories for general τ $(\tau\colon H \to G)$

τ-graded category

An H-Hom-graded category $\mathcal{C}$ such that

1. $\mathcal{C}^1$ is G-graded
2. $\mathrm{Hom}^h(X, Y) = 0$ for homogeneous X, Y unless $|Y| = \tau(h)|X|$.

Example: $\mathrm{Vect}^\bullet_H$ is τ-graded for any G.

Choose each $|\Bbbk_h| = \tau(h)$

$\implies \mathrm{Hom}^h(\Bbbk_a, \Bbbk_{a'}) = 0$ unless $a' = ha$, hence $\tau(a') = \tau(h)\tau(a)$.

Semisimplicity

$(\tau\colon H \to G)$

Shifts

A H-Hom-graded category $\mathcal{C}$ **has shifts** if for each $h \in H$, $X \in \mathcal{C}$ there is an object $X\langle h\rangle$ together with a canonical degree h isomorphism

$$r_{X,h}\colon X \to X\langle h\rangle\,.$$

Semisimple H-Hom-graded category

An H-Hom-graded category with shifts such that the 1-subcategory is semisimple.

Semisimplicity

$(\tau\colon H \to G)$

Shifts and semisimplicity for a general τ-graded category

As for the underlying H-Hom-graded category.

Example: $\mathrm{Vect}_H^{fd,\bullet}$

- Simple objects $(\Bbbk_h)_{h\in H}$ have $\mathrm{Hom}(\Bbbk_h, \Bbbk_{h'}) = \delta_{h'h}\Bbbk$.
- Shifts $\Bbbk_h\langle h'\rangle = \Bbbk_{h'h}$
  with $r_{\Bbbk_h,h'}\colon \bigoplus_{a\in H} \delta_{ah} v \mapsto \bigoplus_{a\in H} \delta_{ah'h} v$.

τ-module categories

Shifts as an H-action $(\tau\colon H \to G)$

Lemma

If $\mathcal{C}$ is an H-Hom-graded category with shifts, then $(\mathcal{C}^1, \alpha, \mu, \epsilon)$ is an H-module category (Vect_H-module) with $\alpha^h X = X\langle h\rangle$ and

$$\begin{array}{ccc} \alpha^h X & \xrightarrow{\alpha^h f} & \alpha^h Y \\ {\scriptstyle r^{-1}_{X,h}}\downarrow & & \uparrow{\scriptstyle r_{Y,h}} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} \alpha^h \alpha^{h'} X & \xrightarrow{(\mu_{h,h'})_X} & \alpha^{hh'} X \\ {\scriptstyle r^{-1}_{\alpha^{h'}X,h}}\downarrow & & \uparrow{\scriptstyle r_{X,hh'}} \\ \alpha^{h'} X & \xrightarrow[r^{-1}_{X,h'}]{} & X \end{array}$$

$$X \xrightarrow{\epsilon_X = r_{X,1}} \alpha^1 X = X\langle 1\rangle$$

Shifts as an H-action

$(\tau : H \to G)$

Lemma

Every H-module category $(\mathcal{D}, \alpha, \mu, \epsilon)$ induces an H-Hom-graded category $\mathcal{D}^\bullet$ with shifts $X\langle h\rangle = \alpha^h X$.

$$\mathrm{Hom}_{\mathcal{D}^\bullet}(X,Y) = \bigoplus_{h\in H} \mathrm{Hom}^h_{\mathcal{D}^\bullet}(X,Y) = \bigoplus_{h\in H} \mathrm{Hom}_{\mathcal{D}}(\alpha^h X, Y)$$

$$\mathrm{id}^\bullet_X = \epsilon_X^{-1} : \alpha^1 X \to X$$

$$r_{X,h} = \mathrm{id}_{\alpha^h X} : \alpha^h X \to \alpha^h X$$

$$\begin{array}{ccccc}
\alpha^{h'h} X & & \xrightarrow{\quad f' \circ^\bullet f \quad} & & Z \\
\downarrow{\scriptstyle (\mu_{h'h})_X^{-1}} & & & & \uparrow{\scriptstyle \mathrm{id}} \\
\alpha^{h'}\alpha^h X & \xrightarrow[\alpha^{h'} f]{} & \alpha^{h'} Y & \xrightarrow[f]{} & Z
\end{array}$$

Shifts as an H-action

$(\tau : H \to G)$

Theorem

There is a 2-equivalence between the 2-categories of

- H*-Hom-graded categories with shifts*
- H*-module categories.*

Theorem

There is a 2-equivalence between the 2-categories of

- *semisimple* H*-Hom-graded categories*
- *semisimple* H*-module categories.*

General τ-module categories

$(\tau\colon H \to G)$

τ-graded category

An H-Hom-graded category $\mathcal{C}$ such that

1. $\mathcal{C}^1$ is G-graded
2. $\mathrm{Hom}^h(X, Y) = 0$ for homogeneous X, Y unless $|Y| = \tau(h)|X|$.

τ-module category

A G-graded H-module category $(\mathcal{D}, \alpha, \mu, \epsilon)$ such that

$$|\alpha^h X| = \tau(h)|X|$$

for homogeneous $X \in \mathcal{D}$.

General τ-module categories

$(\tau\colon H \to G)$

Theorem

There is a 2-equivalence between the 2-categories of

- τ*-graded categories with shifts*
- τ*-module categories.*

Theorem

There is a 2-equivalence between the 2-categories of

- *semisimple* τ*-graded categories*
- *semisimple* τ*-module categories.*

Structure theorem

Structure theorem for H-module categories $(\tau\colon H \to G)$

Theorem

An H-module category $\mathcal{D}$ is semisimple if and only if it is of the form

$$\mathcal{D} \simeq \bigoplus_{i \in I} \mathcal{M}(L_i, \psi_i)\,.$$

See e.g. Example 7.4.10 of EGNO Tensor Categories (2015).

Structure theorem for H-Hom-graded categories $(\tau\colon H \to G)$

$\mathcal{M}^\bullet(L,\psi)$ for $L \leq H$, normalised 2-cocycle $\psi\colon H^2 \to \mathrm{Fun}(H/L, \Bbbk^\times)$

The indecomposable semisimple H-Hom-graded category with

- simple objects $(\Bbbk_{hL})_{hL \in H/L}$
- morphisms $\mathrm{Hom}^k(\Bbbk_{hL}, \Bbbk_{h'L}) \cong \delta_{khL,h'L}\Bbbk$
- shifts $\Bbbk_{hL}\langle k\rangle = \Bbbk_{khL}$ with $r_{\Bbbk_{hL},k}$ given by a choice of basis
- composition $r_{\Bbbk_{khL},k'} \circ r_{\Bbbk_{hL},k} = \psi(k',k)(hL)^{-1} r_{\Bbbk_{hL},k'k}$

Example: $\mathrm{Vect}^\bullet_H \simeq \mathcal{M}^\bullet(1,1)$

Structure theorem for H-Hom-graded categories $(\tau\colon H \to G)$

Theorem

An H-Hom-graded category $\mathcal{C}$ is semisimple if and only if it is of the form

$$\mathcal{C} \simeq \bigoplus_{i\in I} \mathcal{M}^\bullet(L_i, \psi_i)\,.$$

- I is in bijection with the simples $(S_i)_{i\in I}$ up to general isomorphism
- $L_i = \{h \in H : S_i\langle h\rangle \cong S_i\}$ *(degree 1 isomorphisms)*

Fix representatives $(h_a)_{a\in J}$ for cosets in H/L_i. For $k \in H$, let h_b represent the coset kh_aL_i, and choose a canonical isomorphism $e^i_{a,k}$ spanning $\mathrm{Hom}^k_{\mathcal{C}}(S_i\langle h_a\rangle, S_i\langle h_b\rangle)$. Then ψ_i is defined by $e^i_{a,k'k} = \psi_i(k',k)(h_aL_i)\cdot(e^i_{b,k'} \circ e^i_{a,k})$.

Structure theorem for general τ-graded categories $(\tau\colon H \to G)$

Lemma

Let $g \in G$, and suppose $L \leq \ker(\tau)$.
Then $\mathcal{M}^\bullet(L,\psi) =: \mathcal{M}^\bullet_\tau(L,\psi)\langle g\rangle$ is τ-graded with each $|\mathbb{k}_{hL}| = \tau(h)g$.

Theorem

A τ-graded category $\mathcal{C}$ is semisimple if and only if it is of the form

$$\mathcal{C} \simeq \bigoplus_{i\in I} \mathcal{M}^\bullet_\tau(L_i,\psi_i)\langle g_i\rangle\,.$$

We choose each L_i and ψ_i as before, and $g_i = |S_i|$.

Conclusion

Outlook

When $\tau =: \chi$ is a crossed module,

- χ-module categories are pseudofunctors $\Bbbk\,\mathrm{Disc}(\mathcal{G}_\chi) \to 2\mathbb{V}\mathrm{ect}$.
- there is a Deligne tensor product $\boxtimes$ of χ-graded categories.

I am working to show that

- $\boxtimes$ corresponds to the Day convolution of χ-module categories.
- finite semisimple χ-graded categories form a fusion 2-category, in which χ-graded fusion categories are the algebra objects.

References

- Kursat Sözer and Alexis Virelizier. *Monoidal Categories Graded by Crossed Modules and 3-Dimensional HQFTs*, Adv. Math 428 (2023). arXiv: 2207.06534

- Etingof, Gelaki, Nikshych and Ostrik. *Tensor Categories*, Mathematical Surveys and Monographs 205 (2015)

Thank you for listening!