Grothendieck-Verdier module categories and Frobenius algebras

Hopf algebras, quantum groups,
monoidal categories
and related structures

ULB

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Plan

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- ☞ **GV categories**
- ☞ **motivation**
- ☞ **GV module categories**
- ☞ **GV Frobenius algebras**
- ☞ **relative Serre functors**

with Gregor Schaumann / Christoph Schweigert / Simon Wood

2306.17668 / 2405.20811

Grothendieck-Verdier categories

DEFINITION — **Grothendieck-Verdier**

dualizing object in a monoidal category $\mathcal{C}$: object K s.t.

- for every $y \in \mathcal{C}$ the functor $x \mapsto \mathrm{Hom}(x \otimes y, K)$ is representable by some $G(y) \in \mathcal{C}$
- $G\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$ is an anti-equivalence
- thus isomorphisms $\varpi_{x,y}\colon \mathrm{Hom}(x \otimes y, K) \xrightarrow{\cong} \mathrm{Hom}(x, Gy)$ natural in $x, y \in \mathcal{C}$

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Grothendieck-Verdier category / *GV-category*:

(also called star-autonomous category)

$=$ monoidal category together with a choice of a dualizing object

Motivation

☞ GV-categories in nature :

- common situation : $\mathcal{C}$ monoidal abelian with non-exact tensor product $\otimes$
- basic example : finite-dimensional bimodules over a finite-dimensional $\Bbbk$-algebra A
  (for generic A : tensor product $\otimes_A$ *right* exact)

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- relevant to HOPF25 : finite-dimensional modules
  over a Hopf algebroid ($\times_R$-Hopf algebra) with bijective antipode
  ALLEN, 2308.01029

note : $\mathcal{C}$ finite tensor category
$\Longrightarrow$ $\mathcal{C}$ monoidally equivalent to modules over a finite-dimensional Hopf algebroid
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  BRUGUIÈRES-LACK-VIRELIZIER, 1003.1920
- relevant to CFT : representations of C_1-cofinite vertex operator algebras
  ALLEN-LENTNER-SCHWEIGERT-WOOD, 2107.05718
- in linear logic : linearly distributive categories with negation
  BARR 1979 ... COCKETT-SEELY 1997 ... PASTRO 2012

A few basic results

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LEMMA — dualizing object is structure

- K dualizing and L invertible $\Longrightarrow$ $K \otimes L$ dualizing
- all dualizing objects are obtained this way BOYARCHENKO-DRINFELD 2013

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☞ G *not* opmonoidal (albeit G^2 monoidal) $\Longrightarrow$ additional structure:

LEMMA — **second tensor product**

- the assignment $x \circledast y := G^{-1}(Gy \otimes Gx)$ defines a functor $\circledast\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
- $\otimes$ right exact $\Longrightarrow$ $\circledast$ provides a left exact monoidal structure on $\mathcal{C}$

BOYARCHENKO-DRINFELD 2013

A few basic results

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☞ two tensor products :

LEMMA — **two monoidal structures**

a GV category $\mathcal{C}$ comes with two monoidal structures :

- $(\mathcal{C}, \otimes, \mathbf{1}, \mathrm{a}^{\otimes}, \mathrm{u}_{\mathrm{l}}^{\otimes}, \mathrm{u}_{\mathrm{r}}^{\otimes})$ right exact
- $(\mathcal{C}, \otimes, K, \mathrm{a}^{\otimes}, \mathrm{u}_{\mathrm{l}}^{\otimes}, \mathrm{u}_{\mathrm{r}}^{\otimes})$ left exact

- for A-bimodules : $\otimes$ = tensor product of comodules over the coalgebra A^*
- in logic : `and` & `or`
- do *not* form a *duoidal* structure

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☞ GV generalizes rigidity :

LEMMA — **GV duality**

- G and G^{-1} are analogous to rigid right / left dualities :

$$\mathrm{Hom}(x, z \otimes Gy) \cong \mathrm{Hom}(x \otimes y, z) \cong \mathrm{Hom}(y, G^{-1}x \otimes z)$$

A few basic results

☞ in particular: mixed multiple tensor products
and thus mixed associativity constraints $=:$ *distributors*:

natural families of morphisms

$$\delta^{\mathrm{l}}_{x,y,z}: \; x \otimes (y \circledast z) \longrightarrow (x \otimes y) \circledast z$$
$$\delta^{\mathrm{r}}_{x,y,z}: \; (x \circledast y) \otimes z \longrightarrow x \circledast (y \otimes z)$$

- in general *not iso*morphisms
- e.g. crucial ingredient in snake identitites for evaluation and coevaluation
- satisfy the $\mathbf{2} + \mathbf{2} + \mathbf{2}$ pentagon identities
  that are valid for the distributors in a linearly distributive category

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- satisfy the $\mathbf{2+2+2}$ pentagon identities
  that are valid for the distributors in a linearly distributive category
  [corollary of properties of *module distributors*]

GV-module categories

GV-module categories

DEFINITION — **GV-module category**

GV-module category over a GV-category $\mathcal{C}$:

module category $(\mathcal{M}, \triangleright, \mathrm{a}^{\otimes}, \mathrm{u}^{\otimes})$ over $(\mathcal{C}, \otimes, 1)$ s.t.

- $c \triangleright - : \mathcal{M} \longrightarrow \mathcal{M}$ has right adjoint for every $c \in \mathcal{C}$
- $- \triangleright m : \mathcal{C} \longrightarrow \mathcal{M}$ has right adjoint for every $m \in \mathcal{M}$

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- reduces to EGNO-definition in case $(\mathcal{C}, \otimes, \mathbf{1})$ is a finite tensor category
- e.g. regular GV-module category $(\mathcal{C}, \otimes)$ with $(c \triangleright -)^{\mathrm{r.a.}} = G^{-1}c \,\otimes\, -$

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PROPOSITION — **induced module structures**

for $(\mathcal{M}, \triangleright)$ a GV-module over $\mathcal{C}$:

$\boxed{c \blacktriangleright - := (Gc \triangleright -)^{\mathrm{r.a.}}}$ is a left $(\mathcal{C}, \circledast)$ module structure $(\mathcal{M}, \blacktriangleright)$

$m \triangleleft c := G^{-1}c \blacktriangleright m$ is right module structure $(\mathcal{M}^{\mathrm{op}}, \triangleleft)$ over $(\mathcal{C}, \otimes)$

$m \blacktriangleleft c := G^{-1}c \triangleright m$ is right module structure $(\mathcal{M}^{\mathrm{op}}, \blacktriangleleft)$ over $(\mathcal{C}, \circledast)$

GV-module functors

☞ module functors : lax, oplax and strong variants

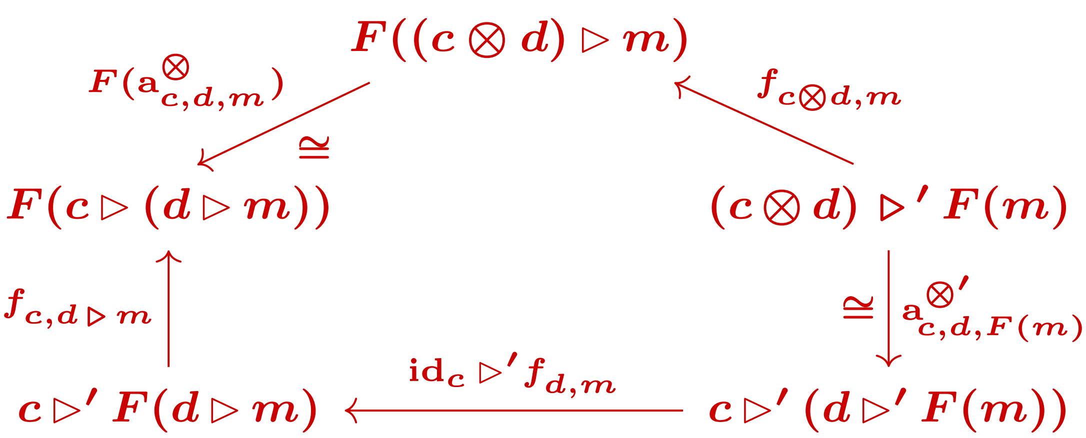
DEFINITION — **module functor**

for $(\mathcal{M}, \triangleright)$ and $(\mathcal{M}', \triangleright')$ left GV-modules over a GV-category $\mathcal{C}$:

lax $\triangleright$*-module functor* from $\mathcal{M}$ to $\mathcal{M}'$ is :

- functor $F\colon \mathcal{M} \to \mathcal{M}'$
- natural family of morphisms $f_{x,m}\colon\, x \triangleright' F(m) \longrightarrow F(x \triangleright m)$ for all $x \in \mathcal{C}$, $m \in \mathcal{M}$

  satisfying pentagon and triangle identities



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$$
\begin{array}{ccccc}
 & & F((c \otimes d) \triangleright m) & & \\
 & \swarrow^{F(\mathrm{a}^{\otimes}_{c,d,m})}_{\cong} & & \searrow^{f_{c\otimes d,m}} & \\
F(c \triangleright (d \triangleright m)) & & & & (c \otimes d) \triangleright' F(m) \\
\downarrow{\scriptstyle f_{c,\,d \triangleright m}} & & & & \cong\,\downarrow{\scriptstyle \mathrm{a}^{\otimes'}_{c,d,F(m)}} \\
c \triangleright' F(d \triangleright m) & & \xrightarrow{\;\mathrm{id}_c \triangleright' f_{d,m}\;} & & c \triangleright' (d \triangleright' F(m))
\end{array}
$$

☞ module functors : lax, oplax and strong variants

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☞ analogous variants involving $(\mathcal{C}, \otimes)$

☞ by definition : for a GV-module $\mathcal{M}$ over a GV-category $\mathcal{C}$

- $\mathrm{R}^{\triangleright}_{m} : \; {}^{\otimes}_{\mathcal{C}}\mathcal{C} \longrightarrow {}_{\mathcal{C}}\mathcal{M} \qquad c \longmapsto c \triangleright m$ is a strong $\triangleright$-module functor
- $\mathrm{R}^{\blacktriangleright}_{m} : \; {}^{\otimes}_{\mathcal{C}}\mathcal{C} \longrightarrow {}_{\mathcal{C}}\mathcal{M} \qquad c \longmapsto c \blacktriangleright m$ is a strong $\blacktriangleright$-module functor

for every $m \in \mathcal{M}$

GV-module functors

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☞ analogous variants involving $(\mathcal{C},\otimes)$

☞ can show : for a GV-module $\mathcal{M}$ over a GV-category $\mathcal{C}$

- $\mathrm{R}^{\triangleright}_m\colon\ {}^{\otimes}_{\mathcal{C}}\mathcal{C}\longrightarrow{}_{\mathcal{C}}\mathcal{M}\qquad c\mapsto c\triangleright m$ is an oplax $\blacktriangleright$-module functor
- $\mathrm{R}^{\blacktriangleright}_m\colon\ {}^{\otimes}_{\mathcal{C}}\mathcal{C}\longrightarrow{}_{\mathcal{C}}\mathcal{M}\qquad c\mapsto c\blacktriangleright m$ is a lax $\triangleright$-module functor

for every $m\in\mathcal{M}$

Module distributors

Module distributors

DEFINITION — **module distributors**

for $\mathcal{M}$ left GV-module category over GV-category $\mathcal{C}$: define

- *right module distributor* $\delta^{\mathrm{r}}_{x,y,m} : (x \otimes y) \triangleright m \longrightarrow x \blacktriangleright (y \triangleright m)$

  via the oplax $\blacktriangleright$-module functor structure of $\mathrm{R}^{\triangleright}_{m}$

- *left module distributor* $\delta^{\mathrm{l}}_{x,y,m} : x \triangleright (y \blacktriangleright m) \longrightarrow (x \otimes y) \blacktriangleright m$

  via the lax $\triangleright$-module f. structure of $\mathrm{R}^{\blacktriangleright}_{m}$ for $x, y \in \mathcal{C}$ and $m \in \mathcal{M}$

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PROPOSITION — **adjoints of GV-module functor structures**

there are bijections between

- GV-module functor structures on a functor F and
- GV-module functor structures on its adjoints (if the adjoints exist)

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there are bijections

F oplax-$\triangleright$	$\longleftrightarrow$	$F^{\mathrm{r.a.}}$ lax-$\triangleright$
F oplax-$\blacktriangleright$	$\longleftrightarrow$	$F^{\mathrm{r.a.}}$ lax-$\blacktriangleright$
F oplax-$\triangleright$	$\longleftrightarrow$	$F^{\mathrm{r.a.}}$ oplax-$\blacktriangleright$
F lax-$\triangleright$	$\longleftrightarrow$	$F^{\mathrm{r.a.}}$ lax-$\blacktriangleright$
F strong-$\triangleright$	$\longleftrightarrow$	$F^{\mathrm{r.a.}}$ strong-$\blacktriangleright$

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PROPOSITION — **adjoints of GV-module functor structures**

there are bijections

$F^{\mathrm{l.a.}}$ oplax-$\triangleright$	$\longleftrightarrow$	F lax-$\triangleright$
$F^{\mathrm{l.a.}}$ oplax-$\blacktriangleright$	$\longleftrightarrow$	F lax-$\blacktriangleright$
$F^{\mathrm{l.a.}}$ oplax-$\triangleright$	$\longleftrightarrow$	F oplax-$\blacktriangleright$
$F^{\mathrm{l.a.}}$ lax-$\triangleright$	$\longleftrightarrow$	F lax-$\blacktriangleright$
$F^{\mathrm{l.a.}}$ strong-$\triangleright$	$\longleftrightarrow$	F strong-$\blacktriangleright$

THEOREM — **pentagons for module distributors**

the module distributors satisfy six pentagon identities :

- $$((x \boldsymbol{\otimes} y) \boldsymbol{\otimes} z) \triangleright m \;\substack{\longrightarrow\,\longrightarrow \\ \longrightarrow\,\longrightarrow\,\longrightarrow}\; x \blacktriangleright (y \blacktriangleright (z \triangleright m))$$
  $$(x \boldsymbol{\otimes} y) \triangleright (z \triangleright m) \;\substack{\longrightarrow\,\longrightarrow \\ \longrightarrow\,\longrightarrow\,\longrightarrow}\; x \blacktriangleright (y \triangleright (z \triangleright m))$$
  for δ^{r}

- $$(x \otimes y) \otimes (z \blacktriangleright m) \;\substack{\longrightarrow\,\longrightarrow \\ \longrightarrow\,\longrightarrow\,\longrightarrow}\; (x \otimes (y \otimes z)) \blacktriangleright m$$
  $$x \triangleright ((y \boldsymbol{\otimes} z) \blacktriangleright m) \;\substack{\longrightarrow\,\longrightarrow \\ \longrightarrow\,\longrightarrow\,\longrightarrow}\; (x \otimes y) \blacktriangleright (z \blacktriangleright m)$$
  for δ^{l}

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  $$(x \boldsymbol{\otimes} y) \triangleright (z \blacktriangleright m) \;\substack{\longrightarrow\,\longrightarrow \\ \longrightarrow\,\longrightarrow\,\longrightarrow}\; x \boldsymbol{\otimes} ((y \otimes z) \blacktriangleright m)$$
  for δ^{r} & δ^{l}

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- $((x \boldsymbol{\otimes} y) \boldsymbol{\otimes} z) \triangleright m \overset{\longrightarrow\longrightarrow}{\longrightarrow\longrightarrow\longrightarrow} x \blacktriangleright (y \blacktriangleright (z \triangleright m))$
  $(x \boldsymbol{\otimes} y) \triangleright (z \triangleright m) \overset{\longrightarrow\longrightarrow}{\longrightarrow\longrightarrow\longrightarrow} x \blacktriangleright (y \triangleright (z \triangleright m))$
  for δ^{r}
- $(x \otimes y) \otimes (z \blacktriangleright m) \overset{\longrightarrow\longrightarrow}{\longrightarrow\longrightarrow\longrightarrow} (x \otimes (y \otimes z)) \blacktriangleright m$
  $x \triangleright ((y \boldsymbol{\otimes} z) \blacktriangleright m) \overset{\longrightarrow\longrightarrow}{\longrightarrow\longrightarrow\longrightarrow} (x \otimes y) \blacktriangleright (z \blacktriangleright m)$
  for δ^{l}
- $(x \triangleright (y \boldsymbol{\otimes} z)) \triangleright m \overset{\longrightarrow\longrightarrow}{\longrightarrow\longrightarrow\longrightarrow} (x \otimes y) \blacktriangleright (z \triangleright m)$
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  for δ^{r} & δ^{l}

☞ conceptually : these diagrams precisely express :

- $\delta^{\mathrm{r}} / \delta^{\mathrm{l}}$ being an oplax / lax $\triangleright$-module structure on $\mathbf{R}_m^{\triangleright}$
- compatibility of the weak module structures on $\mathbf{R}_m^{\triangleright}$ and of those on $\mathbf{R}_m^{\blacktriangleright}$

Module distributors

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  for δ^{r}

- $(x \otimes y) \otimes (z \blacktriangleright m) \;\substack{\longrightarrow\longrightarrow \\ \longrightarrow\longrightarrow\longrightarrow}\; (x \otimes (y \otimes z)) \blacktriangleright m$

  $x \triangleright ((y \boldsymbol{\otimes} z) \blacktriangleright m) \;\substack{\longrightarrow\longrightarrow \\ \longrightarrow\longrightarrow\longrightarrow}\; (x \otimes y) \blacktriangleright (z \blacktriangleright m)$

  for δ^{l}

- $(x \triangleright (y \boldsymbol{\otimes} z)) \triangleright m \;\substack{\longrightarrow\longrightarrow \\ \longrightarrow\longrightarrow\longrightarrow}\; (x \otimes y) \blacktriangleright (z \triangleright m)$

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  for δ^{r} & δ^{l}

☞ for GV-*bi*module categories : in addition mixed module distributors

$$\delta^{\mathrm{l}}_{c,m,d} : \quad c \triangleright (m \blacktriangleleft d) \longrightarrow (c \triangleright m) \blacktriangleleft d$$

$$\delta^{\mathrm{r}}_{c,m,d} : \quad (c \blacktriangleright m) \triangleleft d \longrightarrow c \blacktriangleright (m \triangleleft d)$$

obeying **32** pentagon identites

Module distributors

THEOREM — **pentagons for module distributors**

the module distributors satisfy six pentagon identities :

- $((x \circledast y) \circledast z) \triangleright m \longrightarrow \longrightarrow x \blacktriangleright (y \blacktriangleright (z \triangleright m))$
  $(x \circledast y) \triangleright (z \triangleright m) \longrightarrow \longrightarrow x \blacktriangleright (y \triangleright (z \triangleright m))$ — for δ^{r}
- $(x \otimes y) \otimes (z \blacktriangleright m) \longrightarrow \longrightarrow (x \otimes (y \otimes z)) \blacktriangleright m$
  $x \triangleright ((y \circledast z) \blacktriangleright m) \longrightarrow \longrightarrow (x \otimes y) \blacktriangleright (z \blacktriangleright m)$ — for δ^{l}
- $(x \triangleright (y \circledast z)) \triangleright m \longrightarrow \longrightarrow (x \otimes y) \blacktriangleright (z \triangleright m)$
  $(x \circledast y) \triangleright (z \blacktriangleright m) \longrightarrow \longrightarrow x \circledast ((y \otimes z) \blacktriangleright m)$ — for δ^{r} & δ^{l}

☞ by specializing to the regular module category :

- the six pentagon identites for the distributors of a GV-category $\mathcal{C}$ are valid
- distributors of $\mathcal{C}$ find their conceptual home as weak module functor structures

GV-Frobenius algebras

GV-Frobenius algebras

☞ recall : two equitable monoidal structures on a GV-category

DEFINITION — **GV-(co)algebras**

- *GV-algebra* in a GV-category $\mathcal{C}$: algebra in $(\mathcal{C}, \otimes, 1, a^{\otimes}, u_l^{\otimes}, u_r^{\otimes})$
- *GV-coalgebra* in a GV-category $\mathcal{C}$: coalgebra in $(\mathcal{C}, \circledast, K, a^{\circledast}, u_l^{\circledast}, u_r^{\circledast})$

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DEFINITION — **GV-Frobenius algebras**

GV-Frobenius algebra in a GV-category $\mathcal{C}$: $(A, \mu, \eta, \Delta, \varepsilon)$ s.t.

- (A, μ, η) GV-algebra in $\mathcal{C}$
- (A, Δ, ε) GV-coalgebra in $\mathcal{C}$
- $(\mu \circledast \mathrm{id}_A) \circ \qquad \circ (\mathrm{id}_A \otimes \Delta)$

$$= \Delta \circ \mu = (\mathrm{id}_A \circledast \mu) \circ \qquad \circ (\Delta \otimes \mathrm{id}_A)$$

☞ recall : two equitable monoidal structures on a GV-category

DEFINITION — **GV-(co)algebras**

- *GV-algebra* in a GV-category $\mathcal{C}$: algebra in $(\mathcal{C}, \otimes, 1, \mathrm{a}^{\otimes}, \mathrm{u}_{\mathrm{l}}^{\otimes}, \mathrm{u}_{\mathrm{r}}^{\otimes})$
- *GV-coalgebra* in a GV-category $\mathcal{C}$: coalgebra in $(\mathcal{C}, \varotimes, K, \mathrm{a}^{\varotimes}, \mathrm{u}_{\mathrm{l}}^{\varotimes}, \mathrm{u}_{\mathrm{r}}^{\varotimes})$

DEFINITION — **GV-Frobenius algebras**

GV-Frobenius algebra in a GV-category $\mathcal{C}$: $(A, \mu, \eta, \Delta, \varepsilon)$ s.t.

- (A, μ, η) GV-algebra in $\mathcal{C}$
- (A, Δ, ε) GV-coalgebra in $\mathcal{C}$
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Egger 2010

☞ standard results for monoidal categories generalize :

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PROPOSITION — **equivalent definitions**

Frobenius algebra structures on a GV-algebra A in a GV-category $\mathcal{C}$ are in bijection with :

- invariant non-degenerate GV-pairings $\kappa_A : A \otimes A \longrightarrow K$
- Frobenius forms $\lambda_A : A \longrightarrow K$

this extends to an equivalence between

- the groupoid of GV-Frobenius algebras in $\mathcal{C}$
- the groupoid of (A, κ_A) -algebras in $\mathcal{C}$
- the groupoid of (A, λ_A) -algebras in $\mathcal{C}$

GV-module categories vs categories of modules

Module cat's vs algebras

☞ not too hard to show :

PROPOSITION — **modules give module categories**

for A a GV-algebra in a GV-category $\mathcal{C}$ that admits equalizers :

- the category mod-A is a left GV-module category over $\mathcal{C}$
- the module distributors of mod-A are the distributors of $\mathcal{C}$

☞ for converse statement need further ingredients

Module cat's vs algebras

DEFINITION — **admissible objects**

for $\mathcal{M}$ a GV-module category over a GV-category $\mathcal{C}$:

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☞ $\widehat{\mathcal{M}}^{\otimes}$ / $\widehat{\mathcal{M}}^{\otimes}$ can be 'small' – even zero

☞ $\widehat{\mathcal{C}}^{\otimes}$ / $\widehat{\mathcal{C}}^{\otimes}$ are *monoidal* subcategories of $\mathcal{C}$

☞ $\widehat{\mathcal{M}}^{\otimes}$ / $\widehat{\mathcal{M}}^{\otimes}$ are naturally module categories over $\widehat{\mathcal{C}}^{\otimes}$ / $\widehat{\mathcal{C}}^{\otimes}$

DEFINITION — $\mathcal{C}$-(co)generator in $\mathcal{M}$

for $\mathcal{M}$ a module category over a monoidal category $\mathcal{C}$:

- $m_\circ \in \mathcal{M}$ a $\mathcal{C}$ *-generator*

  $:\Longleftrightarrow$ for every $m \in \mathcal{M}$ there is an epimorphism $c \triangleright m_\circ \longrightarrow m$ for some $c \in \mathcal{C}$

☞ dually : $n_\circ \in \mathcal{M}$ a $\mathcal{C}$ *-cogenerator*

Module cat's vs algebras

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  the objects $m \in \mathcal{M}$ for which mod-$\underline{\mathbf{Hom}}(m,m) \simeq \mathcal{M}$ as $(\mathcal{C},\otimes)$-module are precisely the $\mathcal{C}$-generators in $\widehat{\mathcal{M}}^{\otimes}$

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Relative Serre functors

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☞ relative Serre functor for $\mathcal{C}$-module $\mathcal{M}$ with respect to $H: \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathcal{C}$:

$S: \mathcal{M} \longrightarrow \mathcal{M}$ with natural isomorphism $H(n, S(m)) \xrightarrow{\cong} D(H(m,n))$

with $D: \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$ a duality functor

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- variant: partially defined relative Serre functor: $S: \widehat{\mathcal{M}} \to \mathcal{M}$ for $\widehat{\mathcal{M}} \subseteq \mathcal{M}$

  $n \in \mathcal{M}$, $m \in \widehat{\mathcal{M}}$

  here: $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}^{\otimes}$ or $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}^{\otimes}$

  $D = G$

  $H = \underline{\mathrm{Hom}}$

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LEMMA — assignments S and $\widetilde{S}$

- $m \in \widehat{\mathcal{M}}^{\otimes} \iff$ there is $S(m) \in \mathcal{M}$ such that
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- can choose $S(m) = \big(\underline{\mathrm{Hom}}(m,-)\big)^{\mathrm{r.a.}}(K)$

- $n \in \widehat{\mathcal{M}}^{\otimes} \iff$ there is $\widetilde{S}(n) \in \mathcal{M}$ such that

  $\underline{\mathrm{coHom}}(n,-) \cong \underline{\mathrm{Hom}}(\widetilde{S}(n),-)$ as $\blacktriangleright$-module functors

- can choose $\widetilde{S}(m) = \big(\underline{\mathrm{coHom}}(m,-)\big)^{\mathrm{l.a.}}(1)$

Relative Serre functors

PROPOSITION — **relative Serre functors for GV-modules**

- the assignments S and $\tilde{S}$ extend to functors

$$S\colon\ \widehat{\mathcal{M}}^{\otimes} \rightarrow \mathcal{M} \qquad \text{for } m \in \widehat{\mathcal{M}}^{\otimes},\ n \in \mathcal{M}$$

and

$$\tilde{S}\colon\ \widehat{\mathcal{M}}^{\otimes} \rightarrow \mathcal{M} \qquad \text{for } m \in \widehat{\mathcal{M}}^{\otimes},\ n \in \mathcal{M}$$

- these are partially defined relative Serre functors :

$$\underline{\mathrm{Hom}}(n, Sm) \xrightarrow{\cong} G(\underline{\mathrm{Hom}}(m, n))$$

and

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THEOREM — **equivalence between $\widehat{\mathcal{M}}^{\otimes}$ and $\widehat{\mathcal{M}}^{\otimes}$**

- S and $\tilde{S}$ provide an equivalence $S\colon\ \mathcal{M}^{\otimes} \xleftrightarrow{\simeq} \widehat{\mathcal{M}}^{\otimes} : \tilde{S}$

(abusing notation)

a few further results :

- ☞ S is a twisted module functor :

$$S_{\mathcal{M}}(c \triangleright m) \xrightarrow{\cong} S_{\mathcal{C}}(c) \blacktriangleright S_{\mathcal{M}}(m) \qquad \text{for } c \in \widehat{\mathcal{C}}^{\otimes},\ m \in \widehat{\mathcal{M}}^{\otimes}$$

- ☞ expression for regular module : $S_{\mathcal{C}}(-) \cong G^2(-) \otimes K$
- ☞ $S_{\mathcal{C}}$ is canonically a monoidal equivalence $\widehat{\mathcal{C}}^{\otimes} \xrightarrow{\simeq} \widehat{\mathcal{C}}^{\otimes}$
- ☞ the functor $- \otimes K : \widehat{\mathcal{C}}^{\otimes} \longrightarrow \widehat{\mathcal{C}}^{\otimes}$ is a monoidal equivalence with inverse $- \otimes 1$
- ☞ any isomorphism $m \xrightarrow{\cong} S(m) \in \widehat{\mathcal{M}}^{\otimes}$ endows the algebra $\underline{\mathrm{Hom}}(m,m)$ with the structure of a Grothendieck-Verdier Frobenius algebra

Outlook

☞ **to be studied**:

- symmetric GV-Frobenius algebras in pivotal GV-categories
- GV-module categories corresponding to symmetric GV-Frobenius algebras
- ribbon GV-categories
- lack of coherence / graphical string calculus

  (for a *three*-dimensional graphical calculus see DEMIRDILEK MSc 2024)
- good notion of center
- full CFTs based on VOAs whose rep category has a non-rigid GV structure
- e.g.: in- vs out- bulk fields / ~~sub~~category with exact tensor product


THANK YOU