$\otimes$-Frobenius functors and exact module categories

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(joint w/ Harshit Yadav)

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Hopf25

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Motivation

[Fuchs, Schweigert '20]
Bulk from boundary in finite CFT...

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What is special about ψ?

Frobenius functor vs. Frobenius monoidal

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Let A be a finite-dimensional $\mathbb{K}$-algebra

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- Frobenius iff $G^{ra} \cong G^{la}$

  [Caenepeel, Militaru, Zhu '97]

  Doi-Hopf Modules...

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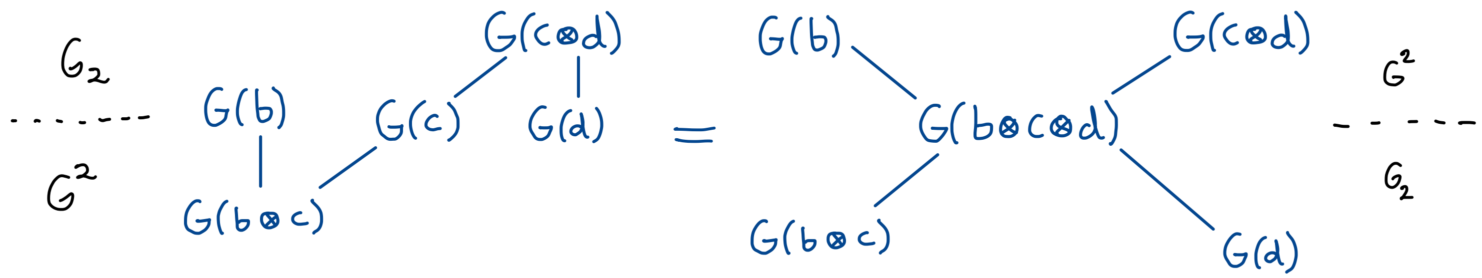

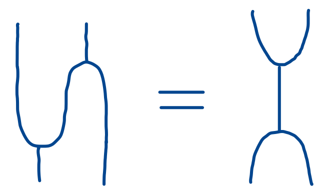

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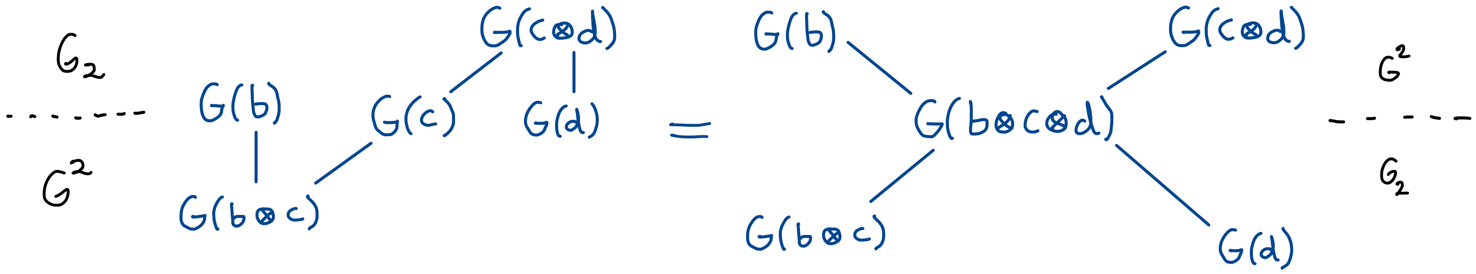

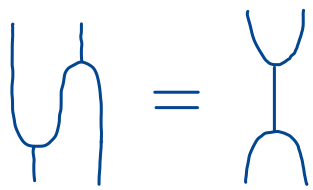

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Proposition

$F : \mathcal{C} \longrightarrow \mathcal{D}$ tensor functor

(i) F^{ra} Frobenius monoidal $\Longrightarrow$ F is Frobenius.

(ii) F^{ra} exact + faithful, F Frobenius $\Longrightarrow$ F^{ra} Frobenius monoidal.

$\otimes$-Frobenius functors

Let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a tensor functor

1. $\mathcal{C}$ acts on $\mathcal{D}$ via $c \triangleright d \triangleleft c' := F(c) \otimes d \otimes F(c')$ $\rightsquigarrow$ $\mathcal{C}$-bimodule category ${}_F\mathcal{D}_F$

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$\rightsquigarrow$ $F^{la}, F^{ra}: {}_F\mathcal{D}_F \longrightarrow \mathcal{C}$ $\mathcal{C}$-bimodule functor [DSPS '14]

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<u>**Proposition**</u> [Flake, Laugwitz, Posur '24]

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(v) F preserves the Radford isomorphism, i.e. $\exists\ \eta : F(D_{\mathcal{C}}) \xrightarrow{\cong} D_{\mathcal{D}}$ s.t.

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<u>Example</u> H fin. dim. Hopf algebra $\exists\ g_H \in G(H)$ $\wedge$ $\alpha_H \in G(H^*)$ s.t.

$$S^4(h) = \alpha_H^{-1} \rightharpoonup (g_H h g_H^{-1}) \leftharpoonup \alpha_H$$ [Radford '76]

$f: H' \longrightarrow H$ Hopf alg. map

$\rightsquigarrow F_f: \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(H')$ is $\otimes$-Frobenius iff $f(g_{H'}) = g_H$ and $\alpha_H \circ f = \alpha_{H'}$

Twisted module categories

Definition $\mathcal{D}$ finite tensor category

(i) Module category: $\mathcal{M}$ + exact action functor $\triangleright : \mathcal{D} \times \mathcal{M} \longrightarrow \mathcal{M}$

(ii) Internal (co)Hom's: $m \in \mathcal{M} \rightsquigarrow -\triangleright m : \mathcal{D} \longrightarrow \mathcal{M}$

$\underline{\mathrm{Hom}}_{\mathcal{M}}(m,-) := (-\triangleright m)^{ra} : \mathcal{M} \longrightarrow \mathcal{D}$ $\qquad$ $\underline{\mathrm{coHom}}_{\mathcal{M}}(m,-) := (-\triangleright m)^{la} : \mathcal{M} \longrightarrow \mathcal{D}$

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$\iff \mathcal{M} \simeq \mathrm{Mod}_A(\mathcal{D})$ where $A = \prod_{i=1}^{n} A_i$ simple

[Coulembier, Stroinski, Zorman '25]

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(iii) Exact module category: $\mathcal{M}$ exact $\Longleftrightarrow$ $\underline{\mathrm{Hom}}_{\mathcal{M}}(-,-)$ is exact

$\Longleftrightarrow \mathcal{M} \simeq \mathrm{Mod}_A(\mathcal{D})$ where $A = \prod_{i=1}^{n} A_i$ simple

[Coulembier, Stroinski, Zorman '25]

(iv) Pivotal module category: $\mathcal{D}$ pivotal, $\mathcal{M}$ pivotal $\Longleftrightarrow$ $\underline{\mathrm{Hom}}_{\mathcal{M}}(m,n) \cong \underline{\mathrm{coHom}}_{\mathcal{M}}(m,n)$

Twisted module categories

Definition $\mathcal{D}$ finite tensor category

(i) Module category: $\mathcal{M}$ + exact action functor $\triangleright : \mathcal{D} \times \mathcal{M} \longrightarrow \mathcal{M}$

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[Coulembier, Stroinski, Zorman '25]

(iv) Pivotal module category: $\mathcal{D}$ pivotal, $\mathcal{M}$ pivotal $\iff \underline{\mathrm{Hom}}_{\mathcal{M}}(m,n) \cong \underline{\mathrm{coHom}}_{\mathcal{M}}(m,n)$

$\iff \mathcal{M} \simeq \mathrm{Mod}_A(\mathcal{D})$, A symmetric Frob. alg. [Shimizu '24]

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[Fuchs, Galindo, J., Schweigert '24] $\qquad$ $\iff \mathcal{M} \simeq \mathrm{Mod}_A(\mathcal{D})$, A symmetric Frob. alg. [Shimizu '24]

3d-TQFTs

(v) Unimodular module category: $\mathcal{M}$ unimodular $\iff \mathrm{Fun}_{\mathcal{D}}(\mathcal{M},\mathcal{M})$ is unimodular.

Twisted module categories

Let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a tensor functor and $\mathcal{M}$ a $\mathcal{D}$-module category

$\rightsquigarrow$ $\mathcal{D}$-module category ${}_F\mathcal{M}$ $\quad c \triangleright^F m := F(c) \triangleright m$

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<u>Theorem</u> [J., Yadav '25]

$F: \mathcal{C} \longrightarrow \mathcal{D}$	$\mathcal{M}$	$F \iff {}_F\mathcal{M}$	F^{ra} preserves
—	exact	perfect iff exact	exact algebras
perfect	unimodular	$\otimes$-Frob. iff unimod.	—
pivotal	pivotal	$\otimes$-Frob. iff pivotal	symmetric Frobenius alg.

Twisted module categories

<u>Example</u> Hopf algebra $\mathcal{U}_q sl_2$ q primitive n^{th}-root of unity

Generators: E, F, K Relations: $E^n = F^n = 0$, $K^n = 1$, $KE = q^2 EK$, $KF = q^{-2} FK$, $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$

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$$\mathbb{K}\mathbb{Z}/n\mathbb{Z} = \langle K \rangle \overset{\iota}{\subset} \mathcal{U}_q\mathfrak{sl}_2 \qquad F_\iota : \mathrm{Rep}(\mathcal{U}_q\mathfrak{sl}_2) \longrightarrow \mathrm{Rep}(\mathbb{K}\mathbb{Z}/n\mathbb{Z})$$

F_ι is Frobenius but not $\otimes$-Frobenius $\left(g_{\mathcal{U}_q\mathfrak{sl}_2} = K^2 \neq g_{\mathbb{K}\mathbb{Z}_n} = 1 \right)$

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semisimple $\longrightarrow$ all unimodular $\left({}^H_L\mathcal{M}_L \text{ unimod}\right)$

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[Shimizu'18] $\longrightarrow$ Only L_n is pivotal over $\Bbbk \mathbb{Z}/n\mathbb{Z}$ $\qquad$ not pivotal over $u_q sl_2$.

Thank you!

Questions?