The ring of differential operators on a monomial curve is a Hopf algebroid.

Myriam Mahaman
joint work with Ulrich Krähmer (arXiv:2405.08490)

Charles University

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Question

Can we find a commutative algebra which is not smooth, yet whose ring of differential operators is a Hopf algebroid?

The ring of differential operators

Motivation

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An $\mathbb{R}$-**linear differential operator** on $\mathbb{R}^n$ is informally defined as an $\mathbb{R}$-linear map

$$D\colon \mathscr{C}^\infty(\mathbb{R}^n) \to \mathscr{C}^\infty(\mathbb{R}^n),$$

which can be expressed using partial derivations, i.e.

$$D = \sum_{0 \leqslant r \leqslant p} f_{i_1,\dots,i_r} \frac{\partial^r}{\partial x_{i_1} \cdots \partial x_{i_r}},$$

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where $f_{i_1,\dots,i_r} \in \mathscr{C}^\infty(\mathbb{R}^n)$.
How can we define such operators in general, when we replace $\mathscr{C}^\infty(\mathbb{R}^n)$ with an arbitrary commutative algebra A?

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As a left A-module,

$$\mathscr{D}_A^1 = A \oplus \operatorname{Der}_k(A),$$

where $\operatorname{Der}_k(A) := \{D \in \operatorname{End}_k(A) \mid \forall a, b \in A \colon D(ab) = aD(b) + D(a)b\}.$

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where $\operatorname{Der}_k(A) := \{D \in \operatorname{End}_k(A) \mid \forall a, b \in A\colon D(ab) = aD(b) + D(a)b\}$.
In general, $\mathscr{D}_A$ is strictly larger than the k-subalgebra generated by A and $\operatorname{Der}_k(A)$.

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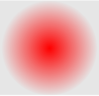
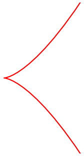
	$k[t]/(t^n)$	$\mathrm{End}_k(A)$
	$k[t^2, t^3]$	generated by t^2, t^3 and operators $D_0 \in \mathscr{D}_A^1$, $L_{-2} \in \mathscr{D}_A^2$, $L_{-3} \in \mathscr{D}_A^3$

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- The unit and counit are fixed:

$$\eta_A \colon A \hookrightarrow \mathscr{D}_A, \qquad \varepsilon_A \colon \mathscr{D}_A \to A, \quad \varepsilon_A(D) := D(1).$$

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- (Sweedler 1974) $\mathscr{D}_A$ is a **(left) bialgebroid** if there exists a comultiplication map

$$\Delta_A \colon \mathscr{D}_A \to \mathscr{D}_A \otimes_A \mathscr{D}_A, \qquad D \mapsto D_{(1)} \otimes_A D_{(2)}$$

which is compatible with the multiplication on $\mathscr{D}_A$.

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- (Schauenburg 1999) $\mathscr{D}_A$ is a **(left) Hopf algebroid** if the Galois map

$$\delta\colon \overline{\mathscr{D}_A} \otimes_A \mathscr{D}_A \to \mathscr{D}_A \otimes_A \mathscr{D}_A, \quad D \otimes_A E \mapsto D_{(1)} \otimes_A D_{(2)} E.$$

is invertible, where $\overline{\mathscr{D}_A} := (\mathscr{D}_A)_A$.

Smooth algebras and Lie-Rinehart algebras

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If $\mathbb{Q} \subseteq A$ and $\mathrm{Der}_k(A)$ is finitely generated projective over A, then

$$\mathscr{D}_A \cong \mathscr{U}_A(\mathrm{Der}_k(A))$$

where $\mathscr{U}_A(\mathrm{Der}_k(A))$ is the **universal enveloping algebra** *of the* **Lie–Rinehart pair** *$(A, \mathrm{Der}_k(A))$.*

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For $D \in \mathrm{Der}_k(A)$, we have

$$\begin{aligned} \varepsilon_A(D) &:= 0, \\ \Delta_A(D) &:= D \otimes_A 1 + 1 \otimes_A D, \\ \delta_A^{-1}(D \otimes_A 1) &:= D \otimes_A 1 - 1 \otimes_A D. \end{aligned}$$

Descent technique

Localizations

Localizations

Theorem (Sweedler 1974, Másson 1991)

Assume:

- *k is a Noetherian commutative ring*
- *A is a commutative algebra essentially of finite type over k,*
- *$A \subseteq K$, where K is a localization of A.*

Then $\mathscr{D}_A$ embeds into $\mathscr{D}_K$, with $\mathscr{D}_K \cong K \otimes_A \mathscr{D}_A$ as a left K-module, and

$$\mathscr{D}_A \cong \{D \in \mathscr{D}_K \mid D(a) \in A \, \forall a \in A\}.$$

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- $\mathscr{D}_K$ is a left Hopf algebroid over K,
- $\mathscr{D}_A \cong \{D \in \mathscr{D}_K \mid D(a) \in A \, \forall a \in A\}$,
- $\mathscr{D}_K \cong K \otimes_A \mathscr{D}_A$ as a left K-module.

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Problems

- Is ι_2 injective?
- $\operatorname{im}(\Delta_K \circ \iota_1) \subseteq \operatorname{im} \iota_2$?

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Definition (Vercruysse 2006)

$\mathscr{D}_A$ is $R(A)$-**locally projective** if for all $D \in \mathscr{D}_A$, there exist $a_1, \ldots, a_n \in A$ and $D_1, \ldots, D_n \in \mathscr{D}_A$ such that

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Suppose $\mathscr{D}_A$ is $R(A)$-locally projective. Then for any embedding of A-modules $\iota\colon X \hookrightarrow Y$, the map $\iota \otimes_A \mathrm{id}_{\mathscr{D}_A}\colon X \otimes_A \mathscr{D}_A \to Y \otimes_A \mathscr{D}_A$ is injective, and

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$$\mathrm{im}(\iota \otimes_A \mathrm{id}_{\mathscr{D}_A}) = \left\{ \sum_i y_i \otimes_A D_i \mid \sum_i D_i(a) y_i \in X \, \forall a \in A \right\}.$$

Descent with local projectivity

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Remark

We obtain a similar diagram for the inverse of the Galois map.

Theorem (Krähmer–M. 2024)

Assume:

- *k is a Noetherian commutative ring*
- *A is a commutative algebra essentially of finite type over k,*
- *$A \subseteq K$, where K is a localization of A for which $\mathscr{D}_K$ is a Hopf algebroid over K.*

If $\mathscr{D}_A$ is $R(A)$-locally projective, then $\mathscr{D}_A$ is a Hopf algebroid over A.

Cocommutativity and Conilpotency

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The Hopf algebroid $\mathscr{U}_A(\mathrm{Der}_K(K))$ is **cocommutative** and it is **conilpotent**, i.e. for any $D \in \ker \varepsilon_K$, there exists $n \geqslant 1$ such that $\overline{\Delta}_K^{(n)}(D) = 0$, where

$$\overline{\Delta}_K(D) := \Delta_K(D) - D \otimes_K 1 - 1 \otimes_K D.$$

Corollary

Under the assumptions of the previous theorem, if $\mathscr{D}_K$ is cocommutative or conilpotent, then so is $\mathscr{D}_A$.

Main result
General setting

Let k be a commutative ring, $A \subseteq K$ be commutative algebras over k, H be a Hopf algebroid over K whose source and target maps are equal, and

$$H(A) := \{h \in H \mid \varepsilon(ha) \in A \, \forall a \in A\}.$$

Theorem (Krähmer–M. 2024)

The Hopf algebroid structure on H descends uniquely to a Hopf algebroid structure over A on $H(A)$ if

- *$H(A)$ is $R(A)$-locally projective, and*
- *the map $\mu \colon K \otimes_A H(A) \to H$, $x \otimes_A h \mapsto xh$ is surjective.*

If H is cocommutative or conilpotent, then so is $H(A)$.

The cusp $A = k[t^2, t^3]$

Ring of differential operators

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$$\mathscr{D}_K = k\langle t, t^{-1}, \partial\rangle / \langle \partial t - t\partial - 1\rangle = \bigoplus_{d\in\mathbb{Z}} t^d k[D_0],$$

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$$\mathscr{D}_A = \bigoplus_{d\in\mathbb{Z}} t^d f_d(D_0) k[D_0]$$

where $f_d = \prod_{i+d=1 \text{ or } i+d<0} (x - i)$.

Theorem (Krähmer–M. 2024)

$\mathscr{D}_A$ *is* $R(A)$*-locally projective.*

For any $D = t^d f(D_0) \in \mathscr{D}_A$, there exists a presentation $D = \sum_{i \in I} D(t^i) D_i$, where

- $I \subseteq \mathbb{Z}$ is a finite set, and
- $D_i := t^{-i} g_i(D_0) \in \mathscr{D}_A$ for each $i \in I$.

Generators

As a k-algebra, $\mathscr{D}_A$ is generated by

$$\begin{aligned} t^2, \quad t^3 &\in \mathscr{D}_A^0, \\ D_0 &\in \mathscr{D}_A^1, \\ L_{-2} := t^{-2} D_0(D_0 - 3) &\in \mathscr{D}_A^2, \\ L_{-3} := t^{-3} D_0(D_0 - 2)(D_0 - 4) &\in \mathscr{D}_A^3. \end{aligned}$$

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In $\mathscr{D}_K \otimes_K \mathscr{D}_K$, the comultiplication yields:

$$\begin{aligned} \Delta_K(D_0) &= D_0 \otimes_K 1 + 1 \otimes_K D_0, \\ \Delta_K(L_{-2}) &= L_{-2} \otimes_K 1 + 2\partial \otimes_K \partial + 1 \otimes_K L_{-2}, \\ \Delta_K(L_{-3}) &= L_{-3} \otimes_K 1 + 3\left(\partial^2 - t^{-1}\partial\right) \otimes_K \partial + 3\partial \otimes_K \left(\partial^2 - t^{-1}\partial\right) + 1 \otimes_K L_{-3}. \end{aligned}$$

Hopf algebroid structure

Theorem (Krähmer–M.)

$\mathscr{D}_A$ is a cocommutative and conilpotent Hopf algebroid over A, with

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\Delta_A(L_{-3}) &= L_{-3} \otimes_A 1 + 3L_{-2} \otimes_A L_{-1} - 3L_{-1} \otimes_A L_{-2}\\
&\quad + 6D_0 \otimes_A (D_0 - 1)L_{-3} - 6L_1 \otimes_A L_{-2}^2 + 1 \otimes_A L_{-3},
\end{aligned}
$$

where $L_1 := tD_0 \in \mathscr{D}_A^1$ and $L_{-1} := t^{-1}D_0(D_0 - 2) \in \mathscr{D}_A^2$.

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\Delta_A(D_0) &= D_0 \otimes_A 1 + 1 \otimes_A D_0, \\
\Delta_A(L_{-2}) &= L_{-2} \otimes_A 1 + 2D_0 \otimes_A (D_0 - 1)L_{-2} - 2L_1 \otimes_A L_{-3} + 1 \otimes_A L_{-2}, \\
\Delta_A(L_{-3}) &= L_{-3} \otimes_A 1 + 3L_{-2} \otimes_A L_{-1} - 3L_{-1} \otimes_A L_{-2} \\
&\quad + 6D_0 \otimes_A (D_0 - 1)L_{-3} - 6L_1 \otimes_A L_{-2}^2 + 1 \otimes_A L_{-3},
\end{aligned}$$

where $L_1 := tD_0 \in \mathscr{D}_A^1$ and $L_{-1} := t^{-1}D_0(D_0 - 2) \in \mathscr{D}_A^2$.
Furthermore, $\mathscr{D}_A$ admits an involutive antipode

$$S(D_0) = -D_0 + 1, \quad S(L_{-2}) = L_{-2}, \quad S(L_{-3}) = -L_{-3}.$$

Conilpotency

The conilpotent filtration of $\mathscr{D}_A$ is not a coalgebra filtration:

$$\overline{\Delta}_A^{(n)}(D) = 0 \text{ if and only if } D \in \mathscr{D}_A^n,$$

but for $D \in \mathscr{D}_A^n$, in general $\Delta_A(D) \notin \sum_{i+j=n} \mathscr{D}_A^i \otimes_A \mathscr{D}_A^j$.

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For example, for $L_{-2} \in \mathscr{D}_A^2$ we have

$$\Delta_A(L_{-2}) = L_{-2} \otimes_A 1 + 2D_0 \otimes_A (D_0 - 1)L_{-2} - 2L_1 \otimes_A L_{-3} + 1 \otimes_A L_{-2},$$

where $D_0, L_1 \in \mathscr{D}_A^1$ and $(D_0 - 1)L_{-2},\ L_{-3} \in \mathscr{D}_A^3$.

Monoidal equivalence

For $A = k[t^2, t^3]$ and $B = k[t]$, we have the $\mathscr{D}_A$-$\mathscr{D}_B$-bimodule

$$\mathscr{D}_K(B, A) := \{D \in \mathscr{D}_K \mid D(b) \in A \,\forall\, b \in B\}.$$

Theorem (Smith-Stafford, Muhasky 1988)

The functor $\mathscr{D}_K(B, A) \otimes_{\mathscr{D}_B} - : \mathscr{D}_B\text{-}\mathbf{Mod} \longrightarrow \mathscr{D}_A\text{-}\mathbf{Mod}$ induces a Morita equivalence.

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If $\mathscr{D}_A$ is a Hopf algebroid, then $\mathscr{D}_A$-**Mod** has a canonical closed monoidal structure:

- A is the monoidal unit,
- $- \otimes_A -$ is the tensor product,
- $\mathrm{Hom}_A(-, -)$ is the inner hom.

Theorem (Krähmer–M. 2024)

The Morita equivalence above is **monoidal**.

Monomial curves

- k is a field of characteristic zero,
- $A := k\left[t^{a_1}, \ldots, t^{a_n}\right]$ where $a_1, \ldots, a_n \in \mathbb{N}$ and $\gcd(a_1, \ldots, a_n) = 1$
  - associated to the **numerical semigroup** $\mathcal{A} := \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$
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Theorem (Krähmer–M. 2024)

- *$\mathscr{D}_A$ is $R(A)$-locally projective.*
- *$\mathscr{D}_A$ is a cocommutative and conilpotent Hopf algebroid over A.*
- *$\mathscr{D}_A$ admits an antipode if $\mathcal{A}$ is* **symmetric**.
- *For any $A = k[t^{a_1}, \ldots, t^{a_n}]$ and $B = k[t^{b_1}, \ldots, t^{b_m}]$, $\mathscr{D}_A$ and $\mathscr{D}_B$ are monoidally Morita equivalent.*

Main results

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Descent theorem for $\mathscr{D}_A$

Assume:

- k is a Noetherian commutative ring
- A is a commutative algebra essentially of finite type over k,
- $A \subseteq K$, where K is a localization of A for which $\mathscr{D}_K$ is a Hopf algebroid over K.

If $\mathscr{D}_A$ is $R(A)$-locally projective, then $\mathscr{D}_A$ is a Hopf algebroid over A.
If $\mathscr{D}_K$ is cocommutative or conilpotent, then so is $\mathscr{D}_A$.

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Example: The cusp

Let k be a field of characteristic zero and $A := k[t^2, t^3]$.
Then $\mathscr{D}_A$ is $R(A)$-locally projective, and is thus a cocommutative and conilpotent Hopf algebroid over A. In particular, it admits an involutive antipode.