The orbit method for the (centerless) Virasoro algebra

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The classical orbit method and the (centerless) Virasoro algebra

The orbit method via Poisson primitive ideal

Prim $\mathrm{U}(W)$ and the Dixmier map for W

The classical orbit method and the (centerless) Virasoro algebra

Lie algebra and universal enveloping algebra

- All vector spaces and Lie algebras are over $\mathbb{C}$.
- Let $\mathfrak{g}$ be a Lie algebra, $\dim \mathfrak{g}$ is finite or countable with ordered basis (e_i).
- The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$

$$\mathrm{U}(\mathfrak{g}) \cong \mathbb{C}\langle e_i \rangle / (e_i e_j - e_j e_i - [e_i, e_j]).$$

Representation theory and primitive spectrum $\operatorname{Prim} \mathrm{U}(\mathfrak{g})$

- Question: Understand $\mathrm{Rep}_{\mathfrak{g}} \equiv \mathrm{Mod}_{\mathrm{U}(\mathfrak{g})}$.
- In general, difficult to understand/classify all simple modules.
- A two-sided ideal Q of $\mathrm{U}(\mathfrak{g})$ is called primitive if Q is the annihilator of a simple module M over $\mathrm{U}(\mathfrak{g})$, i.e.

$$Q = \{x \in \mathrm{U}(\mathfrak{g}) | xM = 0\}.$$

- Refined question: Understand the primitive spectrum $\operatorname{Prim} \mathrm{U}(\mathfrak{g}) = \{\text{primitive ideals } Q \triangleleft \mathrm{U}(\mathfrak{g})\}$ instead.
- $\operatorname{Prim} \mathrm{U}(\mathfrak{g})$ is still hard to understand, so we seek some kinds of correspondence.

The orbit method and Dixmier map

- Let G be the adjoint group of $\mathfrak{g}$ acting on $\mathfrak{g}^*$ by coadjoint action. We denote the space of orbits by $\mathfrak{g}^*/G$.

Theorem (Conze, Dixmier, Duflo, Mathieu, Rentschler)

If $\mathfrak{g}$ is a finite dimensional solvable Lie algebra, then there exists a bijection

$$\overline{\mathrm{Dx}} : \mathfrak{g}^*/G \xrightarrow{\overline{\mathrm{Dx}}} \operatorname{Prim} \mathrm{U}(\mathfrak{g}),$$

$$\chi \mapsto Q_\chi.$$

The Witt and Virasoro algebra

- The centerless Virasoro (Witt) algebra $W = \mathbb{C}[t, t^{-1}]\partial$ is the Lie algebra of derivations on $\mathbb{C}[t, t^{-1}]$, where $\partial = \frac{d}{dt}$, with bracket

$$[f\partial, g\partial] = (fg' - f'g)\partial.$$

- W has a nice countable basis $(e_i = t^{i+1}\partial | i \in \mathbb{Z})$, and

$$[e_i, e_j] = (j - i)e_{i+j}.$$

- The Virasoro algebra Vir is the unique nontrivial central extension of W; it is important in physics, conformal field theory and vertex algebras.
- Problem: W and Vir do not have an adjoint group. Need a slightly different approach.

The orbit method via Poisson primitive ideal

The symmetric algebra $\mathrm{S}(\mathfrak{g})$

- A Poisson algebra A is a commutative algebra with a Poisson bracket (that is a Lie bracket) such that

$$\{xy, z\} = \{x, z\}y + \{y, z\}x.$$

- The symmetric algebra $\mathrm{S}(\mathfrak{g}) \cong \mathbb{C}[e_i]$ is a Poisson algebra with $\{x, y\} = [x, y]$, where $x, y \in \mathfrak{g}$.
- $\mathrm{S}(\mathfrak{g})$ is the associated graded of $\mathrm{U}(\mathfrak{g})$.

Poisson primitive spectrum of $\mathsf{S}(\mathfrak{g})$

- $\mathsf{S}(\mathfrak{g})$ satisfies (generalized) Nullstellensatz

$$\chi \in \mathfrak{g}^* \leftrightarrow \text{ maximal ideal } \mathfrak{m}_\chi = \ker(ev_\chi : \mathsf{S}(\mathfrak{g}) \to \mathbb{C}) \triangleleft \mathsf{S}(\mathfrak{g}).$$

- $I \triangleleft \mathsf{S}(\mathfrak{g})$ is a Poisson ideal if $\{I, \mathsf{S}(\mathfrak{g})\} \subseteq I$.
- A Poisson primitive ideal $P(\chi)$ of $\mathsf{S}(\mathfrak{g})$ is the maximal Poisson ideal contained in $\mathfrak{m}_\chi$.
- The Poisson primitive spectrum of $\mathsf{S}(\mathfrak{g})$:

$$\mathrm{P.Prim}\,\mathsf{S}(\mathfrak{g}) = \{P(\chi) | \chi \in \mathfrak{g}^*\}.$$

- We have a canonical map

$$\mathfrak{g}^* \xrightarrow{\text{P.Core}} \mathrm{P.Prim}\,\mathsf{S}(\mathfrak{g}); \quad \chi \mapsto P(\chi).$$

The orbit method via Poisson primitive ideal

Theorem (Goodearl 2010)

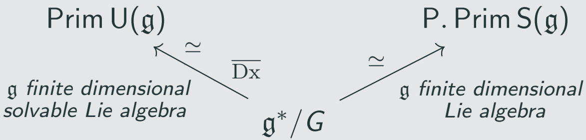
$\operatorname{Prim} \mathrm{U}(\mathfrak{g})$ $\qquad\qquad$ $\mathrm{P.}\operatorname{Prim} \mathrm{S}(\mathfrak{g})$

$\operatorname{Prim} \mathrm{U}(\mathfrak{g}) \xleftarrow[\overline{\mathrm{Dx}}]{\simeq} \mathfrak{g}^*/G$

$\mathfrak{g}$ finite dimensional solvable Lie algebra

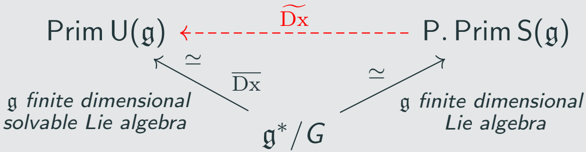
The orbit method via Poisson primitive ideal

Theorem (Goodearl 2010)



The orbit method via Poisson primitive ideal

Theorem (Goodearl 2010)



- The algebraic group G is absent from homeomorphism $\widetilde{\mathrm{Dx}} : \mathrm{P.}\operatorname{Prim} \mathrm{S}(\mathfrak{g}) \to \operatorname{Prim} \mathrm{U}(\mathfrak{g})$ (when $\mathfrak{g}$ is finite-dimensional solvable Lie algebra), thus can be applied in a more general setting (e.g. quantum groups).
- Goal: Study $\operatorname{Prim} \mathrm{U}(W)$ and construct $\widetilde{\mathrm{Dx}} : \mathrm{P.}\operatorname{Prim} \mathrm{S}(W) \to \operatorname{Prim} \mathrm{U}(W)$.

Poisson primitve spectrum of $\mathsf{S}(W)$

- Goal: Study $\operatorname{Prim} \mathsf{U}(W)$ and construct $\widetilde{\mathrm{Dx}} : \mathrm{P}.\operatorname{Prim} \mathsf{S}(W) \to \operatorname{Prim} \mathsf{U}(W)$.

erra, 2023 $P(\chi) \neq 0$ iff χ is a local function on W. They also characterized when $P(\chi) = P(\eta)$ for $\chi, \eta \in W^*$.

- A one-point local function $\chi_{x;\alpha_0,\dots,\alpha_n}$ on W:

$$\chi_{x;\alpha_0,\dots,\alpha_n} : W \to \mathbb{C}; \quad f\partial \mapsto \alpha_0 f(x) + \cdots + \alpha_n f^{(n)}(x),$$

where $x, \alpha_n \neq 0$. n is called the order of χ.

- A local function $\chi = \chi_1 + \cdots + \chi_\ell$ on W is a finite-sum of one-point local function χ_i, where $x_i \neq x_j$.

$\operatorname{Prim} \mathrm{U}(W)$ and the Dixmier map for W

Recipe to construct primitive ideals from local function

We construct representations of W following the orbit method.

- Let χ be a local function on W.
- A polarization $\mathfrak{p}$ of χ is maximal subspace $\mathfrak{p}$ of W such that $\chi\big|_{[\mathfrak{p},\mathfrak{p}]} = 0$ and is also a subalgebra.
- So χ induces a one-dimensional representation of $\mathfrak{p}$, which we write $\mathbb{C}_\chi$. The induced module

$$M_{\chi,\mathfrak{p}} = \mathsf{U}(W) \otimes_{\mathsf{U}(\mathfrak{p})} \mathbb{C}_\chi$$

  is called a local representation of W.
- Let $Q_{\chi,\mathfrak{p}} = \mathrm{Ann}_{\mathsf{U}(W)} M_{\chi,\mathfrak{p}}$.

Dixmier map for the Witt algebra

Theorem (P., 2025)

Let χ be a local function on W. There always exists a polarization $\mathfrak{p}$ of χ. Moreover, $Q_\chi = Q_{\chi,\mathfrak{p}}$ is independent of the choice of polarization $\mathfrak{p}$.

All Q_χ are (completely prime) primitive ideals of $\mathrm{U}(W)$. Thus, we now have a well-defined map

$$W^* \xrightarrow{\mathrm{Dx}} \operatorname{Prim} \mathrm{U}(W), \qquad \begin{aligned} \chi \text{ local} &\mapsto Q_\chi, \\ \chi \text{ not local} &\mapsto 0. \end{aligned}$$

Dixmier map for the Witt algebra

Thus we have a candidate for the Dixmier map

$$\begin{array}{ccc} W^* & \xrightarrow{\text{P.Core}} & \mathsf{P.\,Prim\,S}(W) \\ & \searrow^{\text{Dx}} & \\ & & \mathsf{Prim\,U}(W). \end{array} \qquad \begin{array}{ccc} \chi \text{ local} & \longmapsto & P(\chi) \\ & \searrow & \\ & & Q_\chi. \end{array}$$

Dixmier map for the Witt algebra

Thus we have a candidate for the Dixmier map

$$\begin{array}{ccc} W^* & \xrightarrow{\text{P.Core}} & \mathsf{P.\,Prim\,S}(W) \\ & {\scriptstyle \mathrm{Dx}} \searrow & \big\downarrow {\scriptstyle \widetilde{\mathrm{Dx}}} \\ & & \mathsf{Prim\,U}(W). \end{array} \qquad \begin{array}{ccc} \chi \text{ local} & \longmapsto & P(\chi) \\ & \searrow & \big\downarrow \\ & & Q_\chi. \end{array}$$

Dixmier map for the Witt algebra

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Theorem (P. 2025)

The Dixmier map $\widetilde{\mathrm{Dx}}$ above is well-defined, i.e.

$$\textit{if } P(\chi) = P(\eta), \textit{ then } Q_\chi = Q_\eta.$$

Proof sketch

- Let $\mathfrak{g}_n = \mathbb{C}\{v_0, ..., v_{n-1}\}$ be a finite-dimensional solvable subquotient of W with $[v_i, v_j] = (j-i)v_{i+j}$ if $i+j \leq n$ and 0 otherwise.
- Let $\mathbb{C}[t, t^{-1}, \partial]$ be the localized Weyl algebra with $\partial t - t\partial = 1$.
- Let $\chi = \chi_{x;\alpha_0,...,\alpha_n}$ be a local function on W.

Proof sketch

Theorem (P. 2025)

There exists a graded ring homomorphism

$$\Psi_n : \mathrm{U}(W) \to T_n = \mathbb{C}[t, t^{-1}, \partial] \otimes_{\mathbb{C}} \mathrm{U}(\mathfrak{g}_n),$$

and $\bar{\chi} \in \mathfrak{g}_n^*$ *and a primitive ideal* $Q_{\bar{\chi}} \in \operatorname{Prim} \mathrm{U}(\mathfrak{g}_n)$ *such that*

$$Q_\chi = \Psi_n^{-1}(\mathbb{C}[t, t^{-1}, \partial] \otimes_{\mathbb{C}} Q_{\bar{\chi}}).$$

Let G_n be the adjoint group of $\mathfrak{g}_n$.

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Let G_n be the adjoint group of $\mathfrak{g}_n$.

$$P(\chi) = P(\eta) \Longrightarrow G_n \cdot \bar{\chi} = G_n \cdot \bar{\eta}$$

$$Q_\chi = Q_\eta \qquad\qquad Q_{\bar{\chi}} = Q_{\bar{\eta}}.$$

Proof sketch

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$$\begin{array}{ccc} P(\chi) = P(\eta) & \Longrightarrow & G_n \cdot \bar{\chi} = G_n \cdot \bar{\eta} \\ & & \Downarrow \\ Q_\chi = Q_\eta & & Q_{\bar{\chi}} = Q_{\bar{\eta}}. \end{array}$$

Proof sketch

Theorem (P. 2025)

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$$\begin{array}{ccc} P(\chi) = P(\eta) & \Longrightarrow & G_n \cdot \bar{\chi} = G_n \cdot \bar{\eta} \\ & & \Downarrow \\ Q_\chi = Q_\eta & \Longleftarrow & Q_{\bar{\chi}} = Q_{\bar{\eta}}. \end{array}$$

Questions and Future Work

We have shown that the map $\widetilde{\mathrm{Dx}} : \mathsf{P.\,Prim\,S}(W) \to \mathsf{Prim\,U}(W)$ is well-defined, and it is indeed an instance of orbit method. The case for *Vir* is analogous.

- Is $\widetilde{\mathrm{Dx}}$ surjective? If it is, then every primitive ideals of $\mathsf{U}(W)$ is completely prime and W satisfies ACC on primitive ideals!
- We know that $\widetilde{\mathrm{Dx}}$ is not injective. But how non-injective is it, i.e. all instances where injectivity fails?
- Is $\widetilde{\mathrm{Dx}}$ a continuous and open map?

Figure 1: Link to arXiv:2504.14670 (released today).