Hopf–Galois extensions in noncommutative differential geometry

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Geometry of Hopf–Galois extensions [Schneider '90]

G-Principal Bundle	**Quantum Principal Bundle**
$$\begin{array}{ccc} P & \xleftarrow{\ \triangleleft\ } & P \times G \\ \big\downarrow & & \big\downarrow \cong \\ M & & P \times_M P \end{array}$$	$$\begin{array}{ccc} A & \xrightarrow{\ \Delta_A\ } & A \otimes H \\ \big\uparrow & & \big\uparrow \cong \\ B & & A \otimes_B A \end{array}$$
$M \cong P/G$	$B = A^{\mathrm{co}H} := \{a \in A \mid \Delta_A(a) = a \otimes 1_H\} \subseteq A$
$$\begin{aligned} P \times G &\to P \times_M P \\ (p, g) &\mapsto (p, p \triangleleft g) \end{aligned}$$	$$\begin{aligned} \chi\colon A \otimes_B A &\to A \otimes H \\ a \otimes_B a' &\mapsto a\Delta_A(a') \end{aligned}$$

Hopf–Galois extensions generalize G-principal bundles. They are **quantum principal bundles**:

- A total space algebra
- B base space algebra
- H structure Hopf algebra

Goal of this talk

Construct differential calculi $\Omega^\bullet(A)$, $\Omega^\bullet(B)$, $\Omega^\bullet(H)$ with

$$\begin{array}{ccc} \Omega^\bullet(A) & \xrightarrow{\Delta_A^\bullet} & \Omega^\bullet(A)\otimes\Omega^\bullet(H) \\ \uparrow & & \\ \Omega^\bullet(B) & & \end{array}$$

such that

- $\Omega^\bullet(B) = \Omega^\bullet(A)^{\mathrm{co}\Omega^\bullet(H)} \subseteq \Omega^\bullet(A)$ is a graded Hopf–Galois extension
- the Atiyah sequence is exact

$$0 \to \mathrm{hor}^1 \hookrightarrow \Omega^1(A) \xrightarrow{\pi_v} \mathrm{ver}^1 \to 0$$

- $\Omega^\bullet(A)$ is braided-commutative w.r.t. a canonical braiding

$$\sigma^\bullet \colon \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A)$$

...and discuss examples!

Some literature

Based on seminal works of **Woronowicz**, **Brzeziński–Majid** and **Schauenburg**

- Woronowicz, S.L.: *Differential calculus on compact matrix pseudogroups (quantum groups)*. Comm. Math. Phys. **122** 1 (1989) 125-170.
- Brzeziński, T. and Majid, S.: *Quantum group gauge theory on quantum spaces*. Comm. Math. Phys. **157** (1993) 591-638.
- Schauenburg, P.: *Differential-Graded Hopf Algebras and Quantum Group Differential Calculi*. J. Algebra **180** (1996) 239-286.

We revisit the quantum principal bundle formalism of **Đurđević**, in particular

- Đurđević, M.: *Geometry of Quantum Principal Bundles II - Extended Version*. Rev. Math. Phys. **9**, 5 (1997) 531-607.
- Đurđević, M.: *Quantum Principal Bundles as Hopf-Galois Extensions*. Preprint arXiv:q-alg/9507022.
- Đurđević, M.: *Quantum Gauge Transformations and Braided Structure on Quantum Principal Bundles*. Preprint arXiv:q-alg/9605010.

A reworked version (including new non-trivial examples) is

- Del Donno, A., Latini, E. and TW: *On the Đurđević approach to quantum principal bundles*. Preprint arXiv:2404.07944.

Canonical braiding

$B = A^{\mathrm{co}H} \subseteq A$ be a QPB (=Hopf–Galois extension), i.e. $\chi\colon A \otimes_B A \xrightarrow{\cong} A \otimes H$.

$\exists$ obvious algebra structure on $A \otimes H$ but **not** on $A \otimes_B A$

However, we can **pull back the tensor product multiplication** on $A \otimes H$ to $A \otimes_B A$:

$$\begin{array}{ccc} (A \otimes_B A) \otimes_B (A \otimes_B A) & \xrightarrow{m_{A\otimes_B A}} & A \otimes_B A \\ {\scriptstyle \chi\otimes_B\chi}\downarrow & & \uparrow{\scriptstyle \chi^{-1}} \\ (A \otimes H) \otimes_B (A \otimes H) & \xrightarrow{m_{A\otimes H}} & A \otimes H \end{array}$$

Namely $m_{A\otimes_B A}((a \otimes_B a') \otimes_B (c \otimes_B c')) = a\sigma(a' \otimes_B c)c'$ with

$$\sigma(a \otimes_B a') := a_0 a' \tau(a_1) = a_0 a' (a_1)^{\langle 1\rangle} \otimes_B (a_1)^{\langle 2\rangle}$$

to which we colloquially refer to as the **Đurđević braiding**.

Above $\tau\colon H \to A \otimes_B A$, $\tau(h) := \chi^{-1}(1 \otimes h) = h^{\langle 1\rangle} \otimes_B h^{\langle 2\rangle}$ is the **translation map**.

Proposition (Đurđević '96)

Let $B := A^{\mathrm{co}H} \subseteq A$ be a Hopf–Galois extension and endow $A \otimes_B A$ with the previous multiplication. Then

- *the Galois map $\chi\colon A \otimes_B A \to A \otimes H$ and the translation map $\tau\colon H \to A \otimes_B A$ are algebra morphisms.*
- *$\sigma\colon A \otimes_B A \to A \otimes_B A$ is an isomorphism in ${}_B\mathcal{M}_B$ with inverse*

$$\sigma^{-1}(a \otimes_B a') = \tau(S^{-1}(a'_1))aa'_0,$$

satisfying the braid relations

$$(\sigma \otimes_B \mathrm{id})(\mathrm{id} \otimes_B \sigma)(\sigma \otimes_B \mathrm{id}) = (\mathrm{id} \otimes_B \sigma)(\sigma \otimes_B \mathrm{id})(\mathrm{id} \otimes_B \sigma).$$

- *the following hexagon relations are satisfied*

$$\sigma \circ (m_A \otimes_B \mathrm{id}) = (\mathrm{id} \otimes_B m_A)(\sigma \otimes_B \mathrm{id})(\mathrm{id} \otimes_B \sigma),$$
$$\sigma \circ (\mathrm{id} \otimes_B m_A) = (m_A \otimes_B \mathrm{id})(\mathrm{id} \otimes_B \sigma)(\sigma \otimes_B \mathrm{id}).$$

- *A is braided-commutative with respect to σ, i.e. $m_A \circ \sigma = m_A$, where m_A denotes the multiplication $A \otimes_B A \to A$.*

Noncommutative differential geometry

Let A be an associative unital algebra over a field $\mathbb{k}$.

Definition (Woronowicz '89)

A **differential calculus** (DC) on A is a differential graded algebra (DGA) $(\Omega^\bullet, \wedge, \mathrm{d})$ such that $\Omega = \bigoplus_{n\geq 0} \Omega^n$ with $\Omega^0 = A$ and for all $n > 0$

$$\Omega^n := \mathrm{span}_{\mathbb{k}}\{a^0 \mathrm{d}^1 \wedge \ldots \wedge \mathrm{d}a^n \mid a^0, a^1, \ldots, a^n \in A\}.$$

If A is a right H-comodule algebra we call a DC $\Omega^\bullet$ on A **right H-covariant** if $\Omega^\bullet \in {}_A\mathcal{M}_A^H$ and $\mathrm{d}\colon \Omega^\bullet \to \Omega^{\bullet+1}$ is right H-colinear.

We call the truncation $\Omega^{\leq 1}$ a **(right H-covariant) first order differential calculus** (FODC).

Proposition

*For every FODC $\Omega^{\leq 1}$ there is a DC $\Omega^\bullet$, the **maximal prolongation**, and every other DC extension of $\Omega^{\leq 1}$ is a DGA quotient of $\Omega^\bullet$. If $\Omega^{\leq 1}$ is right H-covariant, so is $\Omega^\bullet$.*

$$\Omega^\bullet := T^{\otimes_A}\Omega^1 \Big/ \Big\langle \sum_i \mathrm{d}a^i \otimes_A \mathrm{d}b^i \mid \sum_i a^i \mathrm{d}b^i = 0 \Big\rangle$$

Graded Hopf algebra

Let $\Omega^1(H)$ bicovariant with maximal prolongation $\Omega^\bullet(H)$.

Lemma (Beggs–Majid '20)

$\Delta\colon H \to H \otimes H$ extends to a morphism of DGAs $\Delta^\bullet\colon \Omega^\bullet(H) \to \Omega^\bullet(H) \otimes \Omega^\bullet(H)$.

$\rightsquigarrow$ write $\Delta^\bullet(\omega) = \omega_{[1]} \otimes \omega_{[2]} \in \bigoplus_{k+\ell=n} \Omega^k(H) \otimes \Omega^\ell(H)$ for $\omega \in \Omega^n(H)$.

Proposition (Schauenburg '96)

$\Omega^\bullet(H)$ is a graded Hopf algebra with

$$\Delta^\bullet\colon \Omega^\bullet(H) \to \Omega^\bullet(H) \otimes \Omega^\bullet(H), \qquad \Delta^\bullet(\omega) = \omega_{[1]} \otimes \omega_{[2]}$$
$$\varepsilon^\bullet\colon \Omega^\bullet(H) \to \Bbbk, \qquad \varepsilon^\bullet(\omega) = 0$$

for all $\omega \in \Omega^\bullet(H)$ with $|\omega| > 0$. The antipode $S^\bullet\colon \Omega^\bullet(H) \to \Omega^\bullet(H)$ is determined by

$$S^\bullet(h^0 \mathrm{d}(h^1) \wedge \ldots \wedge \mathrm{d}(h^k)) = \mathrm{d}(S(h^k)) \wedge \ldots \wedge \mathrm{d}(S(h^1)) S(h^0)$$

for all $h^0 \mathrm{d}(h^1) \wedge \ldots \wedge \mathrm{d}(h^k) \in \Omega^k(H)$.

From now on let $B = A^{\mathrm{co}H} \subseteq A$ be a QPB (=Hopf–Galois extension) and $\Omega^\bullet(H)$ the maximal prolongation of a bicovariant FODC and denote

$$\Lambda^\bullet := {}^{\mathrm{co}H}\Omega^\bullet(H) := \{\omega \in \Omega^\bullet(H) \mid \lambda_\Omega(\omega) = 1 \otimes \omega\}.$$

Definition ([Đurđević '97])

A DC $\Omega^\bullet(A)$ on A is called **complete** if the right H-coaction $\Delta_A\colon A \to A \otimes H$ extends to a morphism

$$\Delta_A^\bullet\colon \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes \Omega^\bullet(H)$$

of DGAs. In this case we refer to $\Omega^\bullet(A)$ as the **total space forms**.

We use the short notation

$$\Delta_A^\bullet(\omega) = \omega_{[0]} \otimes \omega_{[1]}$$

Proposition

Given a complete DC $\Omega^\bullet(A)$ there is another complete DC $\mathrm{ver}^\bullet$ on A defined by $\mathrm{ver}^\bullet := A \otimes \Lambda^\bullet$ with wedge product and differential determined by

$$(a \otimes \vartheta) \wedge (a' \otimes \vartheta') := aa'_0 \otimes S(a'_1)\vartheta a'_2 \wedge \vartheta',$$
$$\mathrm{d}_v(a \otimes \vartheta) := a \otimes \mathrm{d}\vartheta + a_0 \otimes S(a_1)\mathrm{d}(a_2) \wedge \vartheta$$

*for all $a \otimes \vartheta, a' \otimes \vartheta' \in \mathrm{ver}^\bullet$. We call $\mathrm{ver}^\bullet$ the **vertical forms** on A.*

Definition

For a complete calculus $\Omega^\bullet(A)$ on a QPB A we define the **horizontal forms** as the preimage

$$\mathrm{hor}^\bullet := (\Delta_A^\bullet)^{-1}(\Omega^\bullet(A) \otimes H)$$

of $\Omega^\bullet(A) \otimes H$ under $\Delta_A^\bullet \colon \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes \Omega^\bullet(H)$.

- $\mathrm{hor}^\bullet$ is a right H-comodule algebra
- but $\mathrm{hor}^\bullet$ is **not** a DGA!

Definition

Let $\Omega^\bullet(A)$ be a complete calculus on a QPB $B = A^{\mathrm{co}H} \subseteq A$. The corresponding **base forms** are defined as the graded subspace

$$\Omega^\bullet(B) := \{\omega \in \Omega^\bullet(A) \mid \Delta_A^\bullet(\omega) = \omega \otimes 1_H\}.$$

- $\Omega^\bullet(B) \subseteq \Omega^\bullet(A)$ is a DG subalgebra
- but $\Omega^\bullet(B)$ **might not** be generated in degree 0: $B\mathrm{d}B \subseteq \Omega^1(B)$
- however in all examples we encounter $\Omega^\bullet(B) \subseteq \Omega^\bullet(A)$ equals the pullback calculus

Theorem

*For any complete calculus $\Omega^\bullet(A)$ on a QPB $B = A^{\mathrm{co}H} \subseteq A$ the **Atiyah sequence***

$$0 \to \mathrm{hor}^1 \hookrightarrow \Omega^1(A) \xrightarrow{\pi_v} \mathrm{ver}^1 \to 0$$

is exact in the category ${}_A\mathcal{M}^H_A$ of right H-covariant A-bimodules.

Given a complete calculus $\Omega^\bullet(A)$ we define

$$\chi^\bullet \colon \Omega^\bullet(A \otimes_B A) \to \Omega^\bullet(A) \otimes \Omega^\bullet(H)$$
$$\omega \otimes_{\Omega^\bullet(B)} \eta \mapsto \omega \Delta^\bullet_A(\eta) = \omega \wedge \eta_{[0]} \otimes \eta_{[1]}$$

Theorem

$\Omega^\bullet(A)$ complete calculus on QPB $B = A^{\mathrm{co}H} \subseteq A$. Then

$$\Omega^\bullet(B) = \Omega^\bullet(A)^{\mathrm{co}\Omega^\bullet(H)} \subseteq \Omega^\bullet(A)$$

*is a **graded Hopf–Galois extension**.*

Recall the **Đurđević braiding** $\sigma\colon A\otimes_B A\to A\otimes_B A$

$$\sigma(a\otimes_B a') := a_0 a'\tau(a_1) = a_0 a'(a_1)^{\langle 1\rangle}\otimes_B (a_1)^{\langle 2\rangle}$$

$\rightsquigarrow$ extends to braiding $\sigma^\bullet\colon \Omega^\bullet(A)\otimes_{\Omega^\bullet(B)}\Omega^\bullet(A)\to\Omega^\bullet(A)\otimes_{\Omega^\bullet(B)}\Omega^\bullet(A)$ via

$$\sigma^\bullet(\omega\otimes_{\Omega^\bullet(B)}\eta) := (-1)^{|\omega_{[1]}||\eta|}\omega_{[0]}\wedge\eta\wedge\tau^\bullet(\omega_{[1]})$$

Proposition

- $\chi^\bullet\colon \Omega^\bullet(A\otimes_B A)\to\Omega^\bullet(A)\otimes\Omega^\bullet(H)$ *and* $\tau^\bullet\colon\Omega^\bullet(H)\to\Omega^\bullet(A\otimes_B A)$ *are DGA morphisms.*
- $\sigma^\bullet\colon \Omega^\bullet(A)\otimes_{\Omega^\bullet(B)}\Omega^\bullet(A)\to\Omega^\bullet(A)\otimes_{\Omega^\bullet(B)}\Omega^\bullet(A)$ *isomorphism in* ${}_{\Omega^\bullet(B)}\mathcal{M}_{\Omega^\bullet(B)}$ *satisfying*
$$\sigma^\bullet_{12}\sigma^\bullet_{23}\sigma^\bullet_{12}=\sigma^\bullet_{23}\sigma^\bullet_{12}\sigma^\bullet_{23}$$
- $\Omega^\bullet(A)$ *is graded braided-commutative:* $\wedge\circ\sigma^\bullet=\wedge$

Definition (Brzeziński–Majid '93)

If $\Omega^1(A)$ is right H-covariant we have a **quantum principal bundle à la Brzeziński–Majid** if the **vertical map**

$$\mathrm{ver}\colon \Omega^1(A) \to A \otimes \Lambda^1, \qquad \mathrm{ver}(a\mathrm{d}_A(a')) = aa'_0 \otimes S(a'_1)\mathrm{d}_H(a'_2)$$

is well-defined & the sequence $0 \to A\mathrm{d}_A(B)A \hookrightarrow \Omega^1(A) \xrightarrow{\mathrm{ver}} A \otimes \Lambda^1 \to 0$ is exact. We call $\Omega^1_{\mathrm{hor}} := A\mathrm{d}_A(B)A$ the **horizontal 1-forms à la Brzeziński–Majid**.

- QPB à la Brzeziński–Majid $\Rightarrow \Omega^1(A)$ is first order complete.
- $\Omega^1(A)$ first order complete $\Rightarrow \Omega^1(A)$ is QPB à la Brzeziński–Majid iff $\Omega^1_{\mathrm{hor}} = \mathrm{hor}^1$.

Lemma

If $\Omega^1(A)$ is first order complete, then the maximal prolongation $\Omega^\bullet(A)$ is complete.

Example: Bicovariant calculi [DelDonno-Latini-TW '24]

Consider the Galois object $\Bbbk = H^{\mathrm{co}H} \subseteq H$.

For any bicovariant FODC $\Omega^1(H)$ with max. prolongation $\Omega^\bullet(H)$ we have that

- $\Omega^\bullet(H)$ is complete w.r.t. $\Delta\colon H \to H \otimes H$.
- horizontal and base forms are trivial, while $\mathrm{ver}^\bullet = \Omega^\bullet(H)$.
- $\chi^\bullet(\omega \otimes \eta) = \omega \wedge \eta_{[1]} \otimes \eta_{[2]}$ is invertible with inverse $\omega \otimes \eta \mapsto \omega \wedge S^\bullet(\eta_{[1]}) \otimes \eta_{[2]}$.

The Đurđević braiding coincides with the Yetter-Drinfel'd braiding

$$\sigma\colon H \otimes H \to H \otimes H, \qquad \sigma(h \otimes g) = h_1 g S(h_2) \otimes h_3$$

which reads

$$\sigma^\bullet(\omega \otimes \eta) = (-1)^{(|\omega_{[2]}|+|\omega_{[3]}|)|\eta|} \omega_{[1]} \wedge \eta \wedge S^\bullet(\omega_{[2]}) \otimes \omega_{[3]}$$

on differential forms $\omega, \eta \in \Omega^\bullet(H)$. Not symmetric in general!

Example: The noncommutative 2-torus

Let $A := \mathcal{O}_\theta(\mathbb{T}^2) := \mathbb{C}[u, u^{-1}, v, v^{-1}]/\langle vu - e^{i\theta} uv \rangle$ for $\theta \in \mathbb{R}$.

It is a right $H := \mathcal{O}(U(1))$-comodule algebra $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix}$ and a faithfully flat Hopf–Galois extension, with coinvarint subalgebra $B := A^{\mathrm{co}H} = \mathrm{span}_{\mathbb{C}}\{(uv)^{\pm k}\}$ and cleaving map $j \colon H \to A$, $\begin{pmatrix} t^k \\ t^{-k} \end{pmatrix} \mapsto \begin{pmatrix} u^k \\ v^k \end{pmatrix}$ for $k \geq 0$.

Define $\Omega^\bullet(A) = A \oplus \underbrace{\Omega^1(A)}_{=\mathrm{span}_A\{\mathrm{d}u, \mathrm{d}v\}} \oplus \underbrace{\Omega^2(A)}_{=\mathrm{span}_A\{\mathrm{d}u \wedge \mathrm{d}v\}}$ and $\Omega^\bullet(H) = H \oplus \underbrace{\Omega^1(H)}_{=t^{\pm k}\mathrm{d}t}$.

Proposition (DelDonno-Latini-TW '24)

$\Omega^\bullet(A)$ is a complete calculus on the noncommutative 2-torus and $\Omega^\bullet(B)$ is the usual pullback calculus.

$$\Omega^1(A) \to \underbrace{\Omega^1(A) \otimes H}_{\Delta_{\Omega^1(A)}} \oplus \underbrace{A \otimes \Omega^1(H)}_{\mathrm{ver}}$$

$$\Omega^2(A) \to \underbrace{\Omega^2(A) \otimes H}_{\Delta_{\Omega^2(A)}} \oplus \underbrace{\Omega^1(A) \otimes \Omega^1(H)}_{\mathrm{ver}^{1,1} \colon \mathrm{d}u \wedge \mathrm{d}v \mapsto -\mathrm{d}(uv) \otimes t^{-1}\mathrm{d}t}$$

Already known in the literature. Well-defined according to our calculations.

Example: The noncommutative 2-torus

The Đurđević braiding $\sigma\colon A \otimes_B A \to A \otimes_B A$ reads

$$\sigma(u \otimes_B f) = ufu^{-1} \otimes_B u, \qquad \sigma(v \otimes_B f) = vfv^{-1} \otimes_B v$$

for all $f \in A$. For generators of differential forms we obtain

$$\sigma^\bullet(\mathrm{d}u \otimes_{\Omega^\bullet(B)} u) = \mathrm{u} \otimes_{\Omega^\bullet(B)} \mathrm{d}u, \qquad \sigma^\bullet(\mathrm{d}v \otimes_{\Omega^\bullet(B)} v) = v \otimes_{\Omega^\bullet(B)} \mathrm{d}v,$$

$$\sigma^\bullet(\mathrm{d}u \otimes_{\Omega^\bullet(B)} v) = e^{-i\theta} v \otimes_{\Omega^\bullet(B)} \mathrm{d}u, \qquad \sigma^\bullet(\mathrm{d}v \otimes_{\Omega^\bullet(B)} u) = e^{i\theta} u \otimes_{\Omega^\bullet(B)} \mathrm{d}v,$$

$$\sigma^\bullet(\mathrm{d}u \otimes_{\Omega^\bullet(B)} \mathrm{d}u) = -\mathrm{d}u \otimes_{\Omega^\bullet(B)} \mathrm{d}u, \qquad \sigma^\bullet(\mathrm{d}v \otimes_{\Omega^\bullet(B)} \mathrm{d}v) = -\mathrm{d}v \otimes_{\Omega^\bullet(B)} \mathrm{d}v,$$

$$\sigma^\bullet(\mathrm{d}u \otimes_{\Omega^\bullet(B)} \mathrm{d}v) = -e^{-i\theta}\mathrm{d}v \otimes_{\Omega^\bullet(B)} \mathrm{d}u, \qquad \sigma^\bullet(\mathrm{d}v \otimes_{\Omega^\bullet(B)} \mathrm{d}u) = -e^{i\theta}\mathrm{d}u \otimes_{\Omega^\bullet(B)} \mathrm{d}v$$

$$\sigma^\bullet(\mathrm{d}u \wedge \mathrm{d}v \otimes_{\Omega^\bullet(B)} u) = e^{i\theta} u \otimes_{\Omega^\bullet(B)} \mathrm{d}u \wedge \mathrm{d}v,$$

$$\sigma^\bullet(\mathrm{d}u \wedge \mathrm{d}v \otimes_{\Omega^\bullet(B)} v) = e^{-i\theta} v \otimes_{\Omega^\bullet(B)} \mathrm{d}u \wedge \mathrm{d}v$$

etc...

In this case $\sigma^\bullet\colon \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A)$ is symmetric!

Example: $\mathcal{O}_q(\mathrm{SU}_2)$ and Podleś sphere

Let $A := \mathcal{O}_q(\mathrm{SU}_2)$ with $H := \mathcal{O}(U(1))$ coaction $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

This is known to be a faithfully flat Hopf–Galois extension with coinvariant subalgebra the Podleś sphere $B := A^{\mathrm{co}H}$.

Define $\Omega^\bullet(A) = A \oplus \underbrace{\Omega^1(A)}_{=\mathrm{span}_A\{e^\pm, e^0\}} \oplus \underbrace{\Omega^2(A)}_{=\mathrm{span}_A\{e^\pm \wedge e^0, e^+ \wedge e^-\}} \oplus \underbrace{\Omega^3(A)}_{=\mathrm{span}_A\{e^+ \wedge e^- \wedge e^0\}}$ and $\Omega^\bullet(H) = H \oplus \Omega^1(H)$ with $\mathrm{d}t \cdot t = q^2 t \mathrm{d}t$.

Proposition (DelDonno-Latini-TW '24)

$\Omega^\bullet(A)$ is a complete calculus on $\mathcal{O}_q(\mathrm{SU}_2)$. $\Omega^\bullet(B)$ usual pullback calculus.

$$\Omega^1(A) \to \underbrace{\Omega^1(A) \otimes H}_{\Delta_{\Omega^1(A)}} \oplus \underbrace{A \otimes \Omega^1(H)}_{\mathrm{ver}}$$

$$\Omega^2(A) \to \underbrace{\Omega^2(A) \otimes H}_{\Delta_{\Omega^2(A)}} \oplus \underbrace{\Omega^1(A) \otimes \Omega^1(H)}_{\mathrm{ver}^{1,1}(e^+\wedge e^-)=\mathrm{ver}^{1,0}(e^+)\mathrm{ver}^{0,1}(e^-)+\mathrm{ver}^{0,1}(e^+)\mathrm{ver}^{1,0}(e^-)}$$

$$\Omega^3(A) \to \underbrace{\Omega^3(A) \otimes H}_{\Delta_{\Omega^3(A)}} \oplus \underbrace{\Omega^2(A) \otimes \Omega^1(H)}_{\mathrm{ver}^{2,1}}$$

Example: $\mathcal{O}_q(\mathrm{SU}_2)$ and Podleś sphere

Đurđević braiding

$$\begin{aligned}
\sigma(\alpha\otimes_B\alpha)&=\alpha\otimes_B\alpha, & \sigma(\alpha\otimes_B\beta)&=q^{-1}\beta\otimes_B\alpha,\\
\sigma(\alpha\otimes_B\gamma)&=q^{-1}\gamma\otimes_B\alpha, & \sigma(\alpha\otimes_B\delta)&=\delta\otimes_B\alpha+(q^{-1}-q)\beta\otimes_B\gamma,\\
\sigma(\beta\otimes_B\beta)&=\beta\otimes_B\beta, & \sigma(\beta\otimes_B\gamma)&=\gamma\otimes_B\beta,\\
\sigma(\beta\otimes_B\delta)&=q^{-1}\delta\otimes_B\beta, & \sigma(\gamma\otimes_B\gamma)&=\gamma\otimes_B\gamma,\\
\sigma(\gamma\otimes_B\delta)&=q^{-1}\delta\otimes_B\gamma, & \sigma(\delta\otimes_B\delta)&=\delta\otimes_B\delta.
\end{aligned}$$

$$\begin{aligned}
&\sigma^\bullet(e^+\otimes_{\Omega^\bullet(B)}e^+)=\sigma^\bullet(e^-\otimes_{\Omega^\bullet(B)}e^-)=0, && \sigma^\bullet(e^0\otimes_{\Omega^\bullet(B)}e^0)=-e^0\otimes_{\Omega^\bullet(B)}e^0,\\
&\sigma^\bullet(e^+\otimes_{\Omega^\bullet(B)}e^-)=-q^{-2}e^-\otimes_{\Omega^\bullet(B)}e^+, && \sigma^\bullet(e^-\otimes_{\Omega^\bullet(B)}e^+)=-q^{2}e^+\otimes_{\Omega^\bullet(B)}e^-,\\
&\sigma^\bullet(e^\pm\otimes_{\Omega^\bullet(B)}e^0)=-q^{\mp 4}e^0\otimes_{\Omega^\bullet(B)}e^\pm,
\end{aligned}$$

$$\sigma^\bullet(e^0\otimes_{\Omega^\bullet(B)}e^\pm)=-e^\pm\otimes_{\Omega^\bullet(B)}e^0+(1-q^{\mp 4})e^0\wedge e^\pm\otimes_{\Omega^\bullet(B)}1,$$

$$\begin{aligned}
\sigma^\bullet(e^\pm\otimes_{\Omega^\bullet(B)}(e^\pm\wedge e^0))&=0,\\
\sigma^\bullet(e^-\otimes_{\Omega^\bullet(B)}(e^+\wedge e^0))&=q^2(e^+\wedge e^0)\otimes_{\Omega^\bullet(B)}e^-,\\
\sigma^\bullet(e^0\otimes_{\Omega^\bullet(B)}(e^-\wedge e^0))&=(e^-\wedge e^0)\otimes_{\Omega^\bullet(B)}e^0,
\end{aligned}$$

etc... The braiding is symmetric on A but not symmetric on $\Omega^\bullet(A)$!

Crossed product algebras and cleft extensions

Definition

Let A be a right H-comodule algebra and $B := A^{\mathrm{co}H}$. We call $B \subseteq A$

- **trivial extension** if $\exists$ convolution invertible comodule algebra morphism $j\colon H \to A$.
- **cleft extension** if $\exists$ convolution invertible comodule morphism $j\colon H \to A$.

$$\big(\text{trivial extensions}\big) \subseteq \big(\text{cleft extensions}\big) \subseteq \big(\text{Hopf–Galois extensions}\big)$$

Theorem (Doi–Takeuchi '86)

$$\big(\textit{cleft extensions}\big) \stackrel{1:1}{\longleftrightarrow} \big(\textit{crossed product algebras}\big)$$

where trivial extensions correspond to smashed product algebras.

B algebra with $\Bbbk$-linear map $\cdot\colon H \otimes B \to B$ such that $h \cdot (bb') = (h_1 \cdot b)(h_2 \cdot b')$ and $h \cdot 1 = \varepsilon(h)1$. $F\colon H \otimes H \to B$ convolution invertible 2-cocycle with values in B.
$\rightsquigarrow$ $B\#_F H$ is **crossed product algebra** with multiplication

$$(b \otimes h)(b' \otimes h') := b(h_1 \cdot b')F(h_2 \otimes h'_1) \otimes h_3 h'_2.$$

Crossed product algebras and cleft extensions

A FODC $\Omega^1(B)$ is called F-**compatible** if

$$h \cdot (b\mathrm{d}b') = (h_1 \cdot b)\mathrm{d}(h_2 \cdot b'), \qquad \mathrm{d} \circ F = 0.$$

Proposition (Sciandra-TW '23)

For every F-compatible FODC $\Omega^1(B)$ and bicov. $\Omega^1(H)$ there is a FODC on $B\#_F H$

$$\Omega^1_\# := (\Omega^1(B) \otimes H) \oplus (B \otimes \Omega^1(H))$$

with appropriate $B\#_F H$-bimodule structure and $\mathrm{d}_\# = \mathrm{d}_B + \mathrm{d}_H$.

Proposition (DelDonno-Latini-TW '24)

The above can be extended to a complete DC $\Omega^n_\# := \bigoplus_{i=0}^n \Omega^i(B) \otimes \Omega^{n-i}(H)$ on the crossed product algebra $B\#_F H$.

- *the base forms are $\Omega^\bullet(B)$, while $\mathrm{ver}^\bullet = B \otimes \Omega^\bullet(H)$ and $\mathrm{hor}^\bullet = \Omega^\bullet(B) \otimes H$.*
- *The Đurđević braiding reads*

$$\begin{aligned}&\sigma((b \otimes h) \otimes_B (b' \otimes h'))\\ &= (b(h_1 \cdot b')F(h_2 \otimes h'_1)(h_3 h'_2 \cdot F^{-1}(S(h_8) \otimes h_9))F(h_4 h'_3 \otimes S(h_7)) \otimes h_5 h'_4 S(h_6)) \otimes_B (1 \otimes h_{10})\end{aligned}$$

References

- ĐURĐEVIĆ, M.: *Geometry of quantum principal bundles I*. Commun. Math. Phys. **175** 3 (1996) 457–520.
- ĐURĐEVIĆ, M.: *Geometry of Quantum Principal Bundles II - Extended Version*. Rev. Math. Phys. **9**, 5 (1997) 531-607.
- ĐURĐEVIĆ, M.: *Quantum Principal Bundles as Hopf-Galois Extensions*. Preprint arXiv:q-alg/9507022.
- ĐURĐEVIĆ, M.: *Quantum Gauge Transformations and Braided Structure on Quantum Principal Bundles*. Preprint arXiv:q-alg/9605010.
- BRZEZIŃSKI, T. AND MAJID, S.: *Quantum group gauge theory on quantum spaces*. Comm. Math. Phys. **157** (1993) 591-638.
- WORONOWICZ, S.L.: *Differential calculus on compact matrix pseudogroups (quantum groups)*. Comm. Math. Phys. **122** 1 (1989) 125-170.
- DEL DONNO, A., LATINI, E. AND WEBER, T.: *On the Đurđević approach to quantum principal bundles*. Preprint arXiv:2404.07944.
- SCIANDRA, A., WEBER, T.: *Noncommutative differential geometry on crossed product algebras*. J. Algebra. **664** (2025) 129–176.

Thank you for your attention!