Tensor ideals and quantum groups

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joint with Pavel Etingof and Victor Ostrik

Tensor ideals

Tensor ideals of abelian type

Quantum groups

Ideals

A preadditive category $\equiv$ a ring with some fixed idempotents

$\rightsquigarrow$ (two-sided) ideals

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$$\mathrm{Ob}(\mathcal{C}/\mathcal{I}) = \mathrm{Ob}(\mathcal{C}) \qquad (\mathcal{C}/\mathcal{I})(X, Y) = \mathcal{C}(X, Y)/\mathcal{I}(X, Y)$$

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If the preadditive category $\mathcal{C}$ is monoidal (with additive $\otimes$): A (two-sided) **tensor ideal** is an ideal $\mathcal{I}$ in $\mathcal{C}$ with

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$\mathcal{C}/\mathcal{I}$: again a monoidal category

Monoidal terminology: (Fix a field **k**)

A **tensor category** over **k** is a **k**-linear monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ such that

1. $\mathcal{C}$ is abelian with objects of finite length;
2. $\mathbf{k} \xrightarrow{\sim} \mathrm{End}(\mathbb{1})$;
3. $(\mathcal{C}, \otimes, \mathbb{1})$ is rigid.

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A **pseudo-tensor category** over $\mathbf{k}$ is a $\mathbf{k}$-linear monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ such that

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Examples:

- $\mathrm{Rep}\, G$ is a (symmetric) tensor category, for an affine group scheme $G/\mathbf{k}$.
- $\mathrm{Tilt}\, G$ is a (symmetric) pseudo-tensor category, for a reductive group $G/\mathbf{k}$ $(\mathrm{char}(\mathbf{k}) = p > 0)$.

Thick tensor ideals

On a pseudo-tensor category $\mathcal{C}$, we have the coarser notion of a 'thick tensor ideal': a collection of objects $\mathbf{I} \subset \mathrm{Ob}\mathcal{C}$ with:

- If $X \in \mathbf{I}$, then $X \otimes Y \in \mathbf{I} \ni Y \otimes X$ for all $Y \in \mathcal{C}$;
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Abelian envelopes

An abelian envelope of a pseudo-tensor category $\mathcal{D}$ is a 'universal' faithful **k**-linear monoidal functor $\mathcal{D} \to \mathcal{C}$ into a tensor category $\mathcal{C}$.

$$\mathrm{Tens}^{\mathrm{faith}}(\mathcal{D}, \mathcal{C}') \simeq \mathrm{Tens}^{\mathrm{exact}}(\mathcal{C}, \mathcal{C}').$$

Examples:

- $\mathcal{C} \xrightarrow{id} \mathcal{C}$ is an abelian envelope, for a tensor category $\mathcal{C}$.
- $\mathrm{Tilt}\, G \to \mathrm{Rep}\, G$ is an abelian envelope.

Example 1:

For $\delta \in \mathbb{C}$, $\mathcal{OB}(\delta)$ is the universal symmetric pseudo-tensor category over $\mathbb{C}$ on one object of categorical dimension δ. If $\delta \notin \mathbb{Z}$, it is a semisimple tensor category.

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- The abelian envelope of $\mathcal{OB}(n)/\mathcal{I}_d$ is $\mathrm{Rep}\, GL(n+d|d)$.
- Also $\mathcal{OB}(n)$ has abelian envelope (Entova-Hinich-Serganova).

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$\leftrightarrow$ Deligne's Theorem: Over $\mathbb{C}$, any symmetric tensor category of moderate growth 'is' $\mathrm{Rep}\, G$ for a (super)group G.

Example 2:

Let $\mathbf{k}$ be an algebraically closed field with $\mathrm{char}(\mathbf{k}) = p > 0$. The (non-zero) tensor ideals in $\mathrm{Tilt}SL_2$ form one chain

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- The pseudo-tensor categories $\mathrm{Tilt}SL_2/\mathcal{I}_n$ have abelian envelopes 'Ver_{p^n}'. ($\sim$ 2020 Benson-Etingof-Ostrik, C.).
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Conjecture (BEO): Every symmetric tensor category of moderate growth is representation category of a group in $\mathrm{Ver}_{p^\infty} = \cup_n \mathrm{Ver}_{p^n}$.

Theorem (CEO): Every Frobenius exact symmetric tensor category of moderate growth is a representation category of a group in Ver_p.

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By general principles, there is a unique maximal tensor ideal $\mathcal{N}$ in $\mathrm{Tilt}\, G$ ($\sim \mathcal{I}_0$ before), and

$$\mathrm{Ver}_p(G) \;=\; \overline{\mathrm{Tilt}\, G} \;:=\; \mathrm{Tilt}\, G/\mathcal{N}$$

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In general, almost nothing is known, but:

1. Ob will be far from a bijection for $\mathrm{Tilt}\,G$ (∞ to 1).
2. Even classifying thick tensor ideals might be intractable.
3. Most tensor ideals will not be 'prime'.

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is almost never a bijection, the maps

$$\begin{tikzcd}
\{\text{tensor ideals of abelian type}\} \arrow[d, hook] \arrow[dr] & \\
\{\text{prime tensor ideals}\} \arrow[r, two heads, "\mathrm{Ob}"] & \{\text{prime thick tensor ideals}\}
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Motivation: understand tensor ideals of abelian type in $\mathrm{Tilt}\, G$.

Example (CEO): With $p = \mathrm{char}(\mathbf{k})$, there are $n+1$ proper thick tensor ideals in $\mathrm{Rep}\, C_{p^n}$, that are all prime.

There are p^n tensor ideals in $\mathrm{Rep}\, C_{p^n}$.

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However, in $\mathrm{Rep}\, C_p^n$:

$$\{\otimes \text{ ideals of ab. type}\} \;=\; \{\text{prime } \otimes \text{ ideals}\} \;=\; \{\text{prime thick } \otimes \text{ ideals}\}.$$

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The tensor ideals of abelian type are the maximal elements in the fibres of

$$\{\text{tensor ideals}\} \overset{\mathrm{Ob}}{\twoheadrightarrow} \{\text{thick tensor ideals}\},$$

and the kernels of

$$\mathrm{Rep}\, C_{p^n} \xrightarrow{\mathrm{Res}} \mathrm{Rep}\, C_{p^m} \twoheadrightarrow \overline{\mathrm{Rep}\, C_{p^m}}.$$

With $\mathrm{char}(\mathbf{k}) = 2$, and $\mathcal{D} = \mathrm{Rep}\, C_2^2$, the map

$$\{\text{tensor ideals of abelian type}\} \stackrel{\mathrm{Ob}}{\to} \{\text{prime thick tensor ideals}\}$$

is realised as

$$(\mathbb{N} \sqcup \{\infty\}) \times \mathbb{P}^1(\mathbf{k}) \,\sqcup\, \{\infty\} \twoheadrightarrow \mathbb{N} \times \mathbb{P}^1(\mathbf{k}) \,\sqcup\, \{\infty\}$$

(C. - Flake 2023)

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There are prime tensor ideals not of abelian type in $\mathrm{Rep}\, C_2^2$.

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We start by proving a more tractable analogue, by replacing reductive groups in positive characteristic by ...

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For the rest of the talk, $\mathbf{k} = \mathbb{C}$, $\zeta \in \mathbb{C}$ is a primitive ℓ-th root of 1 (with ℓ odd, $\ell > h$, ...) and

$$\mathcal{D} = \mathrm{Tilt}\, U_\zeta(\mathfrak{g}) \;\subset\; \mathrm{Rep}\, U_\zeta(\mathfrak{g}).$$

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Thick tensor ideals were classified by Ostrik. Support-theoretic interpretation (Ostrik, Bezrukavnikov, Boe - Kujawa - Nakano):

Via $H^{2\bullet}(\mathbf{u}_\zeta(\mathfrak{g}), \mathbb{C}) \simeq \mathbb{C}[\mathcal{N}]$ (Ginzburg - Kumar), the support of $T \in \mathrm{Tilt}\, U_\zeta(\mathfrak{g})$ is a G-set in $\mathcal{N}$.

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The assignment

$$I \mapsto \{T \in \mathrm{Tilt}\, U_\zeta(\mathfrak{g}) \mid \mathrm{supp}(T) \subset I\}$$

is an isomorphism between the poset of ideals in $\mathcal{N}/G$ and the poset of thick tensor ideals in $\mathrm{Tilt}\, U_\zeta(\mathfrak{g})$.

The assignment

$$\mathcal{O} \mapsto \mathbf{I}_{\mathcal{O}} := \{T \in \mathrm{Tilt}\, U_\zeta(\mathfrak{g}) \mid \mathcal{O} \not\subset \mathrm{supp}(T)\}$$

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is an isomorphism between the poset $\mathcal{N}/G$ and the poset of **prime** thick tensor ideals in $\mathrm{Tilt}\, U_\zeta(\mathfrak{g})$.

Theorem (CEO, 2025)

The poset of tensor ideals in $\mathrm{Tilt}\, U_\zeta(\mathfrak{g})$ of abelian type is isomorphic to the poset $\mathcal{N}/G$.
Here, $\mathcal{O}$ corresponds to $\mathcal{I}_{\mathcal{O}}$, the maximal element in $\mathrm{Ob}^{-1}(\mathbf{I}_{\mathcal{O}})$, and

$$\{\otimes \text{ ideals of ab. type}\} \stackrel{1:1}{\leftrightarrow} \{\text{prime} \otimes \text{id.}\} \stackrel{1:1}{\leftrightarrow} \{\text{prime thick} \otimes \text{id.}\}.$$

A word about the proof.

Theorem (CEO, 2025)

Let $\mathcal{A}$ be a braided pseudo-tensor category and assume that $\mathcal{R}(\mathcal{A})^M = 0$ for some $M \in \mathbb{N}$. Then

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How to prove that every prime tensor ideal is of abelian type?
($\sim$*Where does the tensor category come from?*)

- For given $\mathcal{O}$, pick a minimal Levi $\mathfrak{l} \subset \mathfrak{g}$ with $\mathcal{O} \cap \mathfrak{l} \neq \varnothing$.
- Then $\mathcal{I}_{\mathcal{O}}$ is the preimage of $\mathcal{I}_{\mathfrak{l}\cap\mathcal{O}}$.
- If $\mathcal{O}$ is distinguished, then $\mathrm{Tilt}\, U_\zeta(\mathfrak{g})/\mathcal{I}_{\mathcal{O}}$ admits an abelian envelope. (e.g $\mathrm{Tilt}\, U_\zeta(\mathfrak{g})/\mathcal{I}_{\mathcal{O}_{reg}} = \overline{\mathrm{Tilt}\, U_\zeta(\mathfrak{g})} \simeq \mathrm{Ver}_q(\mathfrak{g})$)

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(II) With an eye towards $\mathrm{Tilt}\, G$, how does the difficulty of classifying "*all tensor ideals*" versus "*all tensor ideals of abelian type*" compare?

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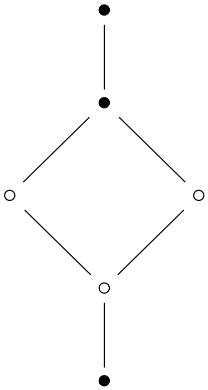
(I) If the situation for reductive groups resembles the quantum case sufficiently, the [BEO] conjecture will not be disproved based on $\mathrm{Tilt}\, G$.

(II) With an eye towards $\mathrm{Tilt}\, G$, how does the difficulty of classifying "*all tensor ideals*" versus "*all tensor ideals of abelian type*" compare?

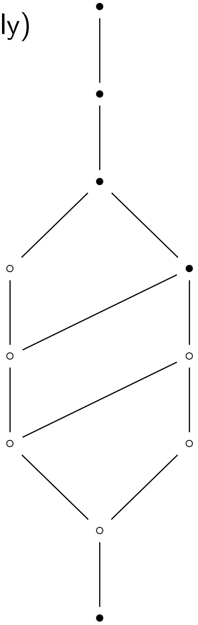
We can visualise with the examples in rank 2: A_2, B_2, C_2, G_2

(A_1 has only two proper tensor ideals, both of abelian type)

Example: A_2



Example: G_2 (Probably)



Example: B_2

The partially ordered set of tensor ideals is infinite in breadth and depth (neither noetherian nor artinian nor interval finite)