Geometry of braidings

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Here

$$c_{12} = c \otimes 1_V : V^{\otimes 3} \to V^{\otimes 3}, \quad c_{23} = 1_V \otimes c : V^{\otimes 3} \to V^{\otimes 3}$$

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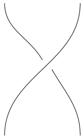
$c =$ 

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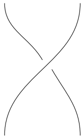
$c =$ 

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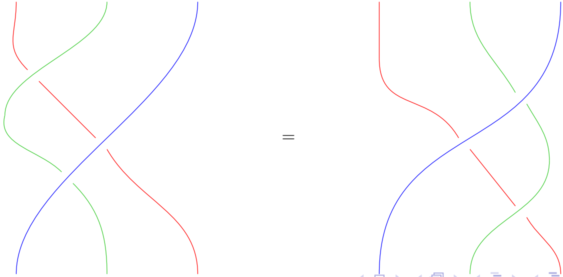

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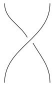
$c =$ 

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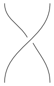
$c =$ 

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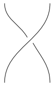
$c =$ 

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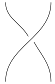
$c =$ 

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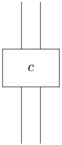

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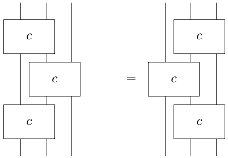

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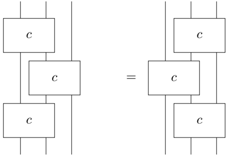
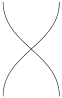


We then just write



for the usual flip map

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This is a less aesthetically pleasing presentation, but it is useful, nonetheless.

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So we get an *algebraic variety* $Z \subseteq \mathbb{A}^{d^4}$ of all possible braidings on V.

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- This is because the scalars c_{kl}^{ij} that we got depend on the choice of basis we have made.
- The group $\mathrm{GL}(V)$ acts on Z by a change of basis, and the $\mathrm{GL}(V)$-orbits correspond to isomorphism types of braidings.

Applications to Nichols algebras

If (V, c) is a braided vector space, then the Nichols algebra $\mathcal{B}(V, c)$ is a graded algebra that plays an important role in the theory of Hopf algebras.

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3. $P(\mathcal{B}(V, c)) = V$.

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Nichols algebras

The following (inaccurate) theorem is (esentially) true:

Theorem

The map

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But what do we actually mean by that? Instead of looking at the Nichols algebra as a whole, let's look at its homogeneous components.

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2. *For every m, the map*

$$\{c | \dim(\mathcal{B}(V,c)_n) = m\} \to Gr(m, V^{\otimes n})$$

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is continuous and $GL(V)$-equivariant.

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Then we can deduce that $\dim(\mathcal{B}(V, c_1)_n) \geq \dim(\mathcal{B}(V, c_2)_n)$ for every n.

Together with Istvan Heckenberger and Leandro Vendramin we proved the following:

Theorem (HMV)

Let $V \in^G_G \mathcal{YD}$ be an irreducible module such that $\langle x | V_x \neq 0 \rangle = G$, and that $\dim(V)$ is prime.

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So it is worth knowing how to describe the invariants in $K[Z]^{\mathrm{GL}(V)}$.

Geometry and invariants

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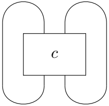

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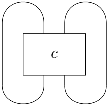

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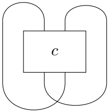

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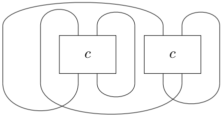

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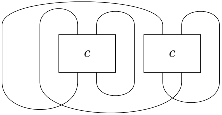

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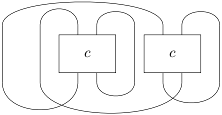

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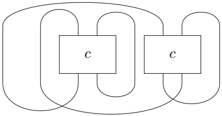

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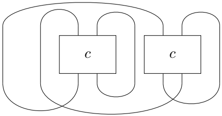

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So this gives us, in principle, a way to describe the variety $Z//\mathrm{GL}(V)$. However, the relations between the invariants can be quite complicated!

Irreducible components

Since Z is an affine variety, one of the very first questions that we can ask is about its **irreducible components**.

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Recall that a braiding $c : V \otimes V \to V \otimes V$ is called of **diagonal type** if has the form

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Let $Y = \overline{Z_{DT}}$. Together with Ben Martin, we proved the following:

Theorem (MM)

- *The closed set Y is an irreducible component of Z.*
- *If char $(K) = 0$ or if char $(K) > \dim(V)$ then we have an isomorphism $K[Y]^{GL(V)} \cong K[q_{ij}^{\pm 1}]^{S_d}$.*

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Examples

For example,

$$\Phi(\mathrm{Tr}(c)) = \sum_i q_{ii}$$

$$\Phi(\mathrm{Tr}(c \circ flip)) = \sum_{i,j} q_{ij}.$$

Characterization of Z_{DT}

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3. *The stabilizer of c in $GL(V)$ contains a maximal torus of $GL(V)$.*

How do we characterize Z_{DT} inside Y?

Theorem (MM)

For $c \in Y$ the following are equivalent:

1. *The orbit $GL(V) \cdot c$ is closed.*
2. $c \in Z_{DT}$.
3. *The stabilizer of c in $GL(V)$ contains a maximal torus of $GL(V)$.*

We do get a concrete family of examples in $Y \backslash Z_{DT}$.

What's next?

Can we generalize this?

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If f is a set theoretical solution of the YBE, it defines a linear solution, where we take $V = KX$ and

$$c(e_x \otimes e_y) = e_z \otimes e_w$$

where $f(x, y) = (z, w)$.

What's next

More generally, given f we can look at solutions of the form

$$c(e_x \otimes e_y) = q_{xy} e_z \otimes e_w,$$

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$$Z_f = \{c | c(e_x \otimes e_y) = q_{xy} e_z \otimes e_w \text{ for some basis } \{e_x\}\}, Y_f = \overline{Z_f}.$$

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Conjecture

The closed subset Y_f is an irreducible component of Z for every set-theoretical solution f.

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With the help of Dejan Govc we proved that Y_f is indeed an irreducible component in this case.
The general case is widely open!