Grothendieck-Verdier categories, Frobenius algebras and relative Serre functors

Christoph Schweigert

Mathematics Department
Hamburg University

based on work with
Jürgen Fuchs, Max Demirdilek, Gregor Schaumann and Simon Wood

September 30, 2025

Overview

1. Grothendieck-Verdier (GV) categories
2. Distributors via module categories, GV Frobenius algebras
3. A three-dimensional graphical calculus

Chapter 1

Grothendieck-Verdier (GV) categories

Rigidity

Definition

1. The right dual of an object c in a monoidal category $\mathcal{C}$ is an object $c^\vee \in \mathcal{C}$, together with evaluation and coevaluation morphisms

$$\mathrm{ev}^r : c^\vee \otimes c \to 1 \qquad \mathrm{coev}^r : 1 \to c \otimes c^\vee$$

such that the usual zig-zag relations hold.

2. A left dual is defined analogously.
3. A category is called rigid, if all its objects have both a left and right dual.

Rigidity

Definition

1. The right dual of an object c in a monoidal category $\mathcal{C}$ is an object $c^\vee \in \mathcal{C}$, together with evaluation and coevaluation morphisms

$$\mathrm{ev}^r : c^\vee \otimes c \to 1 \qquad \mathrm{coev}^r : 1 \to c \otimes c^\vee$$

such that the usual zig-zag relations hold.

2. A left dual is defined analogously.
3. A category is called rigid, if all its objects have both a left and right dual.

Rigidity and the existence of a rigid dual is a property.

Examples

1. Finite-dimensional vector spaces;
finitely generated projective modules over a finite-dimensional k-algebra.
2. Finite-dimensional modules over a finite-dimensional Hopf algebra over a field.
3. Category of tangles, cobordisms.

Consequences of rigidity

Some consequences of rigidity:

- The tensor product of an abelian rigid tensor category is exact.
- Projective objects form a monoidal ideal.
- The dual is opmonoidal,

$$(x \otimes y)^\vee \cong y^\vee \otimes x^\vee$$

Consequences of rigidity

Some consequences of rigidity:

- The tensor product of an abelian rigid tensor category is exact.
- Projective objects form a monoidal ideal.
- The dual is opmonoidal,

$$(x \otimes y)^{\vee} \cong y^{\vee} \otimes x^{\vee}$$

In real life, rigidity is not generic:

Examples

- A a finite-dimensional k-algebra, A-bimod the category of finite-dimensional A-bimodules. Then $B \otimes_A -$, i.e. tensoring with an A-bimodule B, is not necessarily exact.
- The category of finite-dimensional modules over a finite-dimensional Hopf algebroid is, in general, not rigid.
- Vertex algebras to which HLZ tensor product theory applies: Example $\mathcal{W}_{2,3}$ with $c = 0$: tensor product is not exact.

A more general duality structure: GV categories

Definition

1. Let $(\mathcal{C}, \otimes, 1, \alpha, l, r)$ be a monoidal category.
   An object $K \in \mathcal{C}$ is called a dualizing object if, for every $Y \in \mathcal{C}$, the functor

   $$X \mapsto \mathrm{Hom}(X \otimes Y, K)$$

   is representable by some object $GY \in \mathcal{C}$ and the contravariant functor $G: \mathcal{C} \to \mathcal{C}$ is an anti-equivalence.

   $$\mathrm{Hom}(X \otimes Y, K) \cong \mathrm{Hom}(X, GY)\,.$$

   G is called the duality functor with respect to K.
2. A Grothendieck-Verdier category, or GV category, is a monoidal category $(\mathcal{C}, \otimes, 1)$ together with a choice of a dualizing object $K \in \mathcal{C}$.

Examples of GV categories

Remarks

1. Symmetric GV categories are known since the seventies as $*$-autonomous categories and correspond to linearly distributive categories with negation.
2. The choice of dualizing object is structure (in contrast to rigid duality)

Examples of GV categories

Remarks

1. Symmetric GV categories are known since the seventies as $*$-autonomous categories and correspond to linearly distributive categories with negation.
2. The choice of dualizing object is structure (in contrast to rigid duality)

Theorem (ALSW 2021)

Let V be a conformal vertex algebra and $\mathcal{C}$ be a category of V-modules to which the HLZ tensor product theory applies. Then $\mathcal{C}$ has a natural structure of a ribbon GV category with the gradewise dual V' as a dualizing object.

Examples of GV categories

Remarks

1. Symmetric GV categories are known since the seventies as $*$-autonomous categories and correspond to linearly distributive categories with negation.
2. The choice of dualizing object is structure (in contrast to rigid duality)

Theorem (ALSW 2021)

Let V be a conformal vertex algebra and $\mathcal{C}$ be a category of V-modules to which the HLZ tensor product theory applies. Then $\mathcal{C}$ has a natural structure of a ribbon GV category with the gradewise dual V' as a dualizing object.

Theorem (Allen 2023)

Let H be a Hopf algebroid with finite-dimensional base algebra R and an invertible antipode S. Then the category of finite-dimensional H-modules is a GV category.
A dualizing object is given by the vector space dual R^* of the base algebra R with expected R-bimodule and H-module structure given by using the antipode.

Bimodules as monoidal categories

k a field, A a finite-dimensional k-algebra, all vector spaces finite-dimensional. One has

$$\mathrm{Hom}_{A\text{-bimod}}(M_1 \otimes_A M_2, A^*) \cong \mathrm{Hom}_{A\text{-bimod}}(M_1, (M_2)^*)$$

In particular, A^* is a dualizing object for A-bimod

Bimodules as monoidal categories

k a field, A a finite-dimensional k-algebra, all vector spaces finite-dimensional.

Facts

1. The tensor product $B \otimes_A \tilde{B}$ of two A-bimodules B and $\tilde{B}$ is a coequalizer

$$B \otimes A \otimes \tilde{B} \rightrightarrows B \otimes \tilde{B} \longrightarrow B \otimes_A \tilde{B}$$

   and thus right exact. Monoidal unit is A.

2. A^* is a coalgebra, any $M \in A\text{-bimod}$ is an A^*-bicomodule. Equalizer

$$B \otimes^A \tilde{B} \longrightarrow B \otimes \tilde{B} \rightrightarrows B \otimes A^* \otimes \tilde{B}$$

   gives left exact tensor product with monoidal unit A^*.

Remark

We have two tensor products, but we are not aiming for the structure duoidal category.

Bimodules as monoidal categories

k a field, A a finite-dimensional k-algebra, all vector spaces finite-dimensional.

The second tensor product

$$B \otimes^A \tilde{B} \longrightarrow B \otimes \tilde{B} \rightrightarrows B \otimes A^* \otimes \tilde{B}$$

is useful:

Facts

- Eilenberg-Watts:
  $F : A_1\text{-mod} \to A_2\text{-mod}$ right exact, then $F(-) \cong F(A_1) \otimes_{A_1} -$
  $H : A_1\text{-mod} \to A_2\text{-mod}$ left exact, then $H(-) \cong H(A_1^*) \otimes^{A_1} -$
- Morphism of algebras $A \to A'$ gives restriction A'-mod $\to$ A-mod
  Left adjoint $=$ Induction is $A' \otimes_A -$
  Right adjoint $=$ Coinduction is $(A')^* \otimes^A -$

Properties of GV categories

Definition

(Right) internal Hom: $\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z))$.
(Right) internal coHom: $\mathrm{Hom}(X, Y \otimes Z) \cong \mathrm{Hom}(\underline{\mathrm{coHom}}(Z, X), Y)$.

Properties of GV categories

Definition

(Right) internal Hom: $\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z))$.
(Right) internal coHom: $\mathrm{Hom}(X, Y \otimes Z) \cong \mathrm{Hom}(\underline{\mathrm{coHom}}(Z, X), Y)$.

Remarks

1. In a GV category, internal Homs

$$\underline{\mathrm{Hom}}(X, Z) \cong G(X \otimes G^{-1}Z)$$

   exist. Thus $\otimes$ is cocontinuous.

2. A second tensor product

$$X \bullet Y := G^{-1}(GY \otimes GX)$$

   with the dualizing object K as a monoidal unit
   has internal coHoms and is thus continuous.

3. Double dual G^2 is monoidal equivalence $\rightsquigarrow$ Notion of pivotal GV category.
4. Notion of braided GV category, ribbon GV category exist.

Remark 1: GV categories and quantum topology

Theorem (Brochier, Müller, Woike, 2020-2023)

1. *Cyclic associative algebras in* $\mathrm{Lex}^f \leftrightarrow$ *pivotal GV categories*
2. *Cyclic framed* E_2*-algebras in* $\mathrm{Lex}^f \leftrightarrow$ *ribbon GV categories.*
3. *Cyclic framed* E_2*-algebra* $\leftrightarrow$ *ansular functor.*
   (in the case in which Lex^f *is the target category, in ribbon GV categories).*
4. *Under precise conditions, the ansular functor can be extended to a modular functor. This conditions holds for modular categories, but also the Feigin-Fuchs boson.*

Remark 2: Hopf algebroids

We have the following lifting theorem for GV structures:

Theorem (Demirdilek, 2025)

Let $\mathcal{C}, \mathcal{D}$ be closed monoidal categories and $U : \mathcal{C} \to \mathcal{D}$ a closed strong monoidal and isomorphism reflecting functor. Suppose that, for $K \in \mathcal{C}$, the object $U(K)$ is a dualizing object in $\mathcal{D}$. Then K is a dualizing object in $\mathcal{C}$.

Remark 2: Hopf algebroids

We have the following lifting theorem for GV structures:

Theorem (Demirdilek, 2025)

Let $\mathcal{C}, \mathcal{D}$ be closed monoidal categories and $U : \mathcal{C} \to \mathcal{D}$ a closed strong monoidal and isomorphism reflecting functor. Suppose that, for $K \in \mathcal{C}$, the object $U(K)$ is a dualizing object in $\mathcal{D}$. Then K is a dualizing object in $\mathcal{C}$.

A Hopf-algebroid H with base R comes with a forgetful functor $U : \ H\text{-}\mathrm{mod}_{fd} \to R\text{-}\mathrm{bimod}$. Thus

Remark 2: Hopf algebroids

We have the following lifting theorem for GV structures:

Theorem (Demirdilek, 2025)

Let $\mathcal{C}, \mathcal{D}$ be closed monoidal categories and $U : \mathcal{C} \to \mathcal{D}$ a closed strong monoidal and isomorphism reflecting functor. Suppose that, for $K \in \mathcal{C}$, the object $U(K)$ is a dualizing object in $\mathcal{D}$. Then K is a dualizing object in $\mathcal{C}$.

A Hopf-algebroid H with base R comes with a forgetful functor $U : \; H\text{-}\mathrm{mod}_{fd} \to R\text{-}\mathrm{bimod}$. Thus

Corollary

Let R be a finite-dimensional algebra over a field k and H and R-Hopf algebroid. Any H-module structure on R^* gives a dualizing object for $H\text{-}\mathrm{mod}_{fd}$.

Chapter 2

Distributors via module categories, Frobenius algebras

Motivations:

- Mathematical goal: understand (and generalize) distributors

$$\delta^r_{c_1,c_2,c_3}: \; c_1 \otimes (c_2 \bullet c_3) \to (c_1 \otimes c_2) \bullet c_3 \quad \text{and} \quad \delta^l_{c_1,c_2,c_3}: (c_1 \bullet c_2) \otimes c_3 \to c_1 \bullet (c_2 \otimes c_3)$$

- Full CFT needs module categories.
- Full CFT needs Frobenius algebras obtained from module categories to describe field objects.

GV module categories

Definition

A left GV module category over a GV category $(\mathcal{C}, \otimes, K)$ is a left module category $(\mathcal{M}, \triangleright)$ over $(\mathcal{C}, \otimes)$ such that all functors

$$- \triangleright m : {}_{\mathcal{C}}\mathcal{C} \to \mathcal{M} \text{ with } c \mapsto c \triangleright m \quad \text{and} \quad c \triangleright - : \mathcal{M} \to \mathcal{M} \text{ with } m \mapsto c \triangleright m$$

admit a right adjoint.

Call $R_c : \mathcal{M} \to \mathcal{M}$ the right adjoint to $c \triangleright -$.

GV module categories

Definition

A left GV module category over a GV category $(\mathcal{C}, \otimes, K)$ is a left module category $(\mathcal{M}, \triangleright)$ over $(\mathcal{C}, \otimes)$ such that all functors

$$- \triangleright m : {}_{\mathcal{C}}\mathcal{C} \to \mathcal{M} \text{ with } c \mapsto c \triangleright m \quad \text{and} \quad c \triangleright - : \mathcal{M} \to \mathcal{M} \text{ with } m \mapsto c \triangleright m$$

admit a right adjoint.

Call $R_c : \mathcal{M} \to \mathcal{M}$ the right adjoint to $c \triangleright -$.

Proposition (Fuchs, Schaumann, S, Wood)

Let $(\mathcal{M}, \triangleright)$ be a left GV module category over a GV category $(\mathcal{C}, \otimes)$. Then the bifunctor

$$\blacktriangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M} \quad \textit{with} \quad c \blacktriangleright m := R_{Gc}(m)$$

is left exact in each variable and defines a left module category structure over $(\mathcal{C}, \bullet)$.

$\Rightarrow$ Existence of inner Homs for the action $\triangleright$ and inner coHoms for the action $\blacktriangleright$.

Lax module functors

Fact

$\mathcal{C}$ k-linear monoidal category, $\mathcal{M}, \mathcal{N}$ left $\mathcal{C}$-modules
$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "G"] \end{tikzcd}$$

1. Canonical bijection

   Oplax $\mathcal{C}$-module functor structures on F $\leftrightarrow$ Lax $\mathcal{C}$-module structures on G

   such that the adjunction φ with components $\varphi_{m,n} : \mathrm{Hom}(F(m), n) \to \mathrm{Hom}(m, G(n))$ is an isomorphism of $\mathcal{C}$-module profunctors.

Lax module functors

Fact

$\mathcal{C}$ k-linear monoidal category, $\mathcal{M}, \mathcal{N}$ left $\mathcal{C}$-modules
$$\begin{tikzcd} \mathcal{M} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{N} \arrow[ll, bend left, "G"] \end{tikzcd}$$

1. Canonical bijection
   Oplax $\mathcal{C}$-module functor structures on $F \leftrightarrow$ Lax $\mathcal{C}$-module structures on G
   such that the adjunction φ with components $\varphi_{m,n} : \mathrm{Hom}(F(m), n) \to \mathrm{Hom}(m, G(n))$ is an isomorphism of $\mathcal{C}$-module profunctors.
2. Apply to the module functor $\mathcal{C} \to \mathcal{M}$ with $c \mapsto c \triangleright m$
   $\Rightarrow$ for all $m \in \mathcal{M}$ the functor $\underline{\mathrm{Hom}}(m, -)$ is a lax module functor
   $\Rightarrow$ coherent morphisms (but not necessarily isomorphisms!)

$$\delta^r_{c,m,n} : c \otimes \underline{\mathrm{Hom}}(m, n) \to \underline{\mathrm{Hom}}(m, c \triangleright n) \qquad \text{for } c \in \mathcal{C} \text{ and } m, n \in \mathcal{M} \; .$$

Lax module functors

Fact

$\mathcal{C}$ k-linear monoidal category, $\mathcal{M}, \mathcal{N}$ left $\mathcal{C}$-modules
$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "G"] \end{tikzcd}$$

1. Canonical bijection
   Oplax $\mathcal{C}$-module functor structures on F $\leftrightarrow$ Lax $\mathcal{C}$-module structures on G
   such that the adjunction φ with components $\varphi_{m,n} : \mathrm{Hom}(F(m), n) \to \mathrm{Hom}(m, G(n))$ is an isomorphism of $\mathcal{C}$-module profunctors.
2. Apply to the module functor $\mathcal{C} \to \mathcal{M}$ with $c \mapsto c \rhd m$
   $\Rightarrow$ for all $m \in \mathcal{M}$ the functor $\underline{\mathrm{Hom}}(m, -)$ is a lax module functor
   $\Rightarrow$ coherent morphisms (but not necessarily isomorphisms!)
   $$\delta^r_{c,m,n} : c \otimes \underline{\mathrm{Hom}}(m, n) \to \underline{\mathrm{Hom}}(m, c \rhd n) \qquad \text{for } c \in \mathcal{C} \text{ and } m, n \in \mathcal{M} .$$
3. Special case: for $\mathcal{M} = \mathcal{C}$ obtain distributors
   $$\begin{aligned} \delta^r_{c_1,c_2,G(c_3)} : \ & c_1 \otimes \underline{\mathrm{Hom}}(c_3, c_2) = c_1 \otimes (c_2 \bullet G(c_3)) \\ & \to (c_1 \otimes c_2) \bullet G(c_3) = \underline{\mathrm{Hom}}(c_3, c_1 \otimes c_2) \end{aligned}$$

   obeying pentagon diagrams

LD categories and LD Frobenius algebras

As a consequence, any GV category is an LD category:

Definition (Cockett-Seely, 1999)

A linearly distributive (LD) category is a category $\mathcal{C}$, together with two monoidal structures $(\otimes, 1)$ and $(\bullet, K)$ and two natural transformations

$$\delta^r : c_1 \otimes (c_2 \bullet c_3) \to (c_1 \otimes c_2) \bullet c_3 \quad \text{and} \quad \delta^l : (c_1 \bullet c_2) \otimes c_3 \to c_1 \bullet (c_2 \otimes c_3)$$

called distributors, satisfying four triangle and six pentagon axioms as the coherence axioms.

LD categories and LD Frobenius algebras

As a consequence, any GV category is an LD category:

Definition (Cockett-Seely, 1999)

A linearly distributive (LD) category is a category $\mathcal{C}$, together with two monoidal structures $(\otimes, 1)$ and $(\bullet, K)$ and two natural transformations

$$\delta^r : c_1 \otimes (c_2 \bullet c_3) \to (c_1 \otimes c_2) \bullet c_3 \quad \text{and} \quad \delta^l : (c_1 \bullet c_2) \otimes c_3 \to c_1 \bullet (c_2 \otimes c_3)$$

called distributors, satisfying four triangle and six pentagon axioms as the coherence axioms.

We are interested in LD Frobenius algebras in an LD category $\mathcal{C}$

Definition

An LD Frobenius algebra in $\mathcal{C}$ consists of an object $A \in \mathcal{C}$, endowed with an algebra structure (μ, η) with respect to $(\otimes, 1)$ and a coalgebra structures (Δ, ϵ) with respect to $(\bullet, K)$ such that

$$(\mu \bullet \mathrm{id}_A) \circ \delta^r_{A,A,A} \circ (\mathrm{id}_A \otimes \Delta) = \Delta \circ \mu = (\mathrm{id}_A \bullet \mu) \circ \delta^l_{A,A,A} \circ (\Delta \otimes \mathrm{id}_A)$$

as morphisms in $\mathrm{Hom}(A \otimes A, A \bullet A)$.

LD categories and LD Frobenius algebras

We are interested in LD Frobenius algebras in an LD category $\mathcal{C}$

Definition

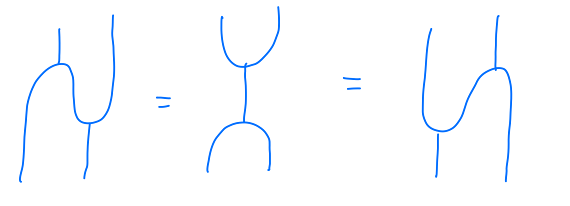
An LD Frobenius algebra in $\mathcal{C}$ consists of an object $A \in \mathcal{C}$, endowed with an algebra structure (μ, η) with respect to $(\otimes, 1)$ and a coalgebra structures (Δ, ϵ) with respect to $(\bullet, K)$ such that

$$(\mu \bullet \mathrm{id}_A) \circ \delta^r_{A,A,A} \circ (\mathrm{id}_A \otimes \Delta) = \Delta \circ \mu = (\mathrm{id}_A \bullet \mu) \circ \delta^l_{A,A,A} \circ (\Delta \otimes \mathrm{id}_A)$$

as morphisms in $\mathrm{Hom}(A \otimes A, A \bullet A)$.

Remark

A tentative graphical representation could be:



LD categories and LD Frobenius algebras

We are interested in LD Frobenius algebras in an LD category $\mathcal{C}$

Definition

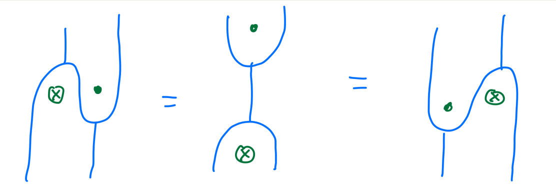
An LD Frobenius algebra in $\mathcal{C}$ consists of an object $A \in \mathcal{C}$, endowed with an algebra structure (μ, η) with respect to $(\otimes, 1)$ and a coalgebra structures (Δ, ϵ) with respect to $(\bullet, K)$ such that

$$(\mu \bullet \mathrm{id}_A) \circ \delta^r_{A,A,A} \circ (\mathrm{id}_A \otimes \Delta) = \Delta \circ \mu = (\mathrm{id}_A \bullet \mu) \circ \delta^l_{A,A,A} \circ (\Delta \otimes \mathrm{id}_A)$$

as morphisms in $\mathrm{Hom}(A \otimes A, A \bullet A)$.

Remark

A tentative graphical representation could be:



LD categories and LD Frobenius algebras

We are interested in LD Frobenius algebras in an LD category $\mathcal{C}$

Definition

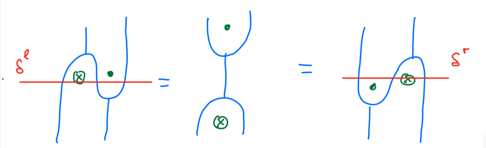
An LD Frobenius algebra in $\mathcal{C}$ consists of an object $A \in \mathcal{C}$, endowed with an algebra structure (μ, η) with respect to $(\otimes, 1)$ and a coalgebra structures (Δ, ϵ) with respect to $(\bullet, K)$ such that

$$(\mu \bullet \mathrm{id}_A) \circ \delta^r_{A,A,A} \circ (\mathrm{id}_A \otimes \Delta) = \Delta \circ \mu = (\mathrm{id}_A \bullet \mu) \circ \delta^l_{A,A,A} \circ (\Delta \otimes \mathrm{id}_A)$$

as morphisms in $\mathrm{Hom}(A \otimes A, A \bullet A)$.

Remark

A tentative graphical representation could be:



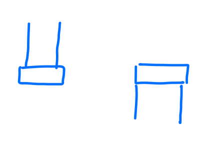
We will see a meaningful graphical calculus later.

LD Frobenius algebras

Proposition

Equivalent characterizations of an LD Frobenius algebra:

1. *Invariant non-degenerate pairing*

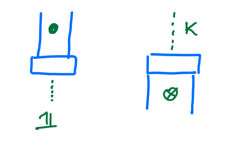


LD Frobenius algebras

Proposition

Equivalent characterizations of an LD Frobenius algebra:

1. *Invariant non-degenerate pairing*

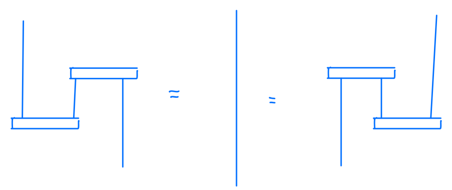


LD Frobenius algebras

Proposition

Equivalent characterizations of an LD Frobenius algebra:

1. *Invariant non-degenerate pairing such that*

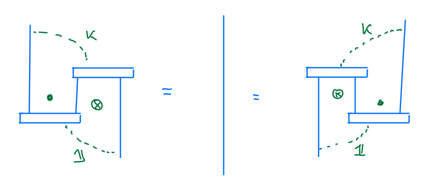


LD Frobenius algebras

Proposition

Equivalent characterizations of an LD Frobenius algebra:

1. *Invariant non-degenerate pairing such that*

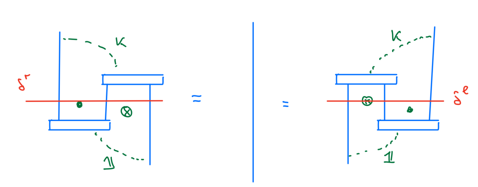


LD Frobenius algebras

Proposition

Equivalent characterizations of an LD Frobenius algebra:

1. *Invariant non-degenerate pairing such that*



LD Frobenius algebras

Proposition

Equivalent characterizations of an LD Frobenius algebra in a GV category:

1. *Invariant non-degenerate pairing*
2. *Frobenius form $\lambda : A \to K$ (defines with LD dualities an isomorphism $A \to G(A)$)*

LD Frobenius algebras

Proposition

Equivalent characterizations of an LD Frobenius algebra in a GV category:

1. *Invariant non-degenerate pairing*
2. *Frobenius form* $\lambda : A \to K$
3. *A left ideal* I *of* A *is a left* A*-module* $\omega : A \otimes I \to I$ *and a monomorphism* $I \xrightarrow{\varphi} A$ *of left modules.*
   We require that for every one-sided ideal I *in* A *that factors through* $\ker \lambda$*,* φ *is the zero morphism.*

LD Frobenius algebras

Proposition

Equivalent characterizations of an LD Frobenius algebra in a GV category:

1. *Invariant non-degenerate pairing*
2. *Frobenius form* $\lambda : A \to K$
3. *A left ideal* I *of* A *is a left* A*-module* $\omega : A \otimes I \to I$ *and a monomorphism* $I \xrightarrow{\varphi} A$ *of left modules.*
   We require that for every one-sided ideal I *in* A *that factors through* $\ker \lambda$, φ *is the zero morphism.*

How to get LD Frobenius algebras?

Towards the existence of Frobenius algebras

Introduce important subcategories of a GV-module category $\mathcal{M}$:

Definition

1. An object $m \in \mathcal{M}$ is called $\otimes$-admissible if
   1. The functor $\underline{\mathrm{Hom}}(m, -)$ is a strong $\triangleright$- module functor.
   2. The functor $\underline{\mathrm{Hom}}(m, -)$ has a right adjoint.
2. $\bullet$-admissible objects are defined using $\underline{\mathrm{coHom}}$.
3. $\widehat{\mathcal{M}}^{\otimes} / \widehat{\mathcal{M}}^{\bullet}$: full subcategories of $\mathcal{M}$ of $\otimes / \bullet$-admissible objects.
4. The subcategories $\widehat{\mathcal{C}}^{\otimes} / \widehat{\mathcal{C}}^{\bullet}$ of $\mathcal{C}$ are obtained by considering $\mathcal{C}$ as a left GV-module category over itself.

Towards the existence of Frobenius algebras

Introduce important subcategories of a GV-module category $\mathcal{M}$:

Definition

1. An object $m \in \mathcal{M}$ is called $\otimes$-admissible if
   1. The functor $\underline{\mathrm{Hom}}(m, -)$ is a strong $\triangleright$- module functor.
   2. The functor $\underline{\mathrm{Hom}}(m, -)$ has a right adjoint.
2. $\bullet$-admissible objects are defined using $\underline{\mathrm{coHom}}$.
3. $\widehat{\mathcal{M}}^{\otimes} / \widehat{\mathcal{M}}^{\bullet}$: full subcategories of $\mathcal{M}$ of $\otimes / \bullet$-admissible objects.
4. The subcategories $\widehat{\mathcal{C}}^{\otimes} / \widehat{\mathcal{C}}^{\bullet}$ of $\mathcal{C}$ are obtained by considering $\mathcal{C}$ as a left GV-module category over itself.

Proposition (Fuchs, Schaumann, S, Wood)

1. *Let $\mathcal{C}$ be a GV category. The subcategories $\widehat{\mathcal{C}}^{\otimes}$ and $\widehat{\mathcal{C}}^{\bullet}$ of $\mathcal{C}$ are unital monoidal subcategories.*
2. *Let $\mathcal{M}$ be a left GV-module category over $\mathcal{C}$. By restriction, the category $\widehat{\mathcal{M}}^{\otimes}$ is a left $\widehat{\mathcal{C}}^{\otimes}$-module category and $\widehat{\mathcal{M}}^{\bullet}$ is a left $\widehat{\mathcal{C}}^{\bullet}$-module category.*

Strong module functors

Lemma

Let A be a finite-dimensional k-algebra, and let ${}_AM_A \in A\text{-}\mathrm{bimod}$ be a finite-dimensional A-bimodule.

The following statements are equivalent:

(i) $\underline{\mathrm{Hom}}(M, -)$ *is a strong module functor.*

(ii) M *has an* $\otimes_A$*-right dual.*

(iii) M_A *is projective as a right A-module.*

(iv) *For all* $X, Y \in A\text{-}\mathrm{bimod}$ *the distributor* $\delta^r \colon\; X \otimes_A (Y \otimes^A M^*) \to\; (X \otimes_A Y) \otimes^A M^*$ *is an isomorphism.*

Partially defined relative Serre functors

Proposition

There exists an equivalence $S\colon \widehat{\mathcal{M}}^{\otimes} \to \widehat{\mathcal{M}}^{\bullet}$ *called the relative Serre functor such that*

$$\underline{\mathrm{Hom}}(m,-) \cong \underline{\mathrm{coHom}}(Sm,-)$$

as an equivalence of $\otimes$*- and* $\bullet$*-module functors.*

Partially defined relative Serre functors

Proposition

There exists an equivalence $S\colon \widehat{\mathcal{M}}^{\otimes} \to \widehat{\mathcal{M}}^{\bullet}$ called the relative Serre functor such that

$$\underline{\mathrm{Hom}}(m,-) \cong \underline{\mathrm{coHom}}(Sm,-)$$

as an equivalence of $\otimes$- and $\bullet$-module functors.

Proposition

Let $\mathcal{C}$ be a GV category and $\mathcal{M}$ a left GV-module category over $\mathcal{C}$.

1. *The relative Serre functor of $\mathcal{C}$ is canonically a monoidal equivalence*

$$S_{\mathcal{C}} : \widehat{\mathcal{C}}^{\otimes} \to \widehat{\mathcal{C}}^{\bullet}$$

2. *The relative Serre functor $S_{\mathcal{M}}$ of $\mathcal{M}$ is a twisted module functor,*

$$S_{\mathcal{M}}(c \triangleright m) \cong S_{\mathcal{C}}(c) \blacktriangleright S_{\mathcal{M}}(m)$$

for $c \in \widehat{\mathcal{C}}^{\otimes}$ and $m \in \widehat{\mathcal{M}}^{\otimes}$.

A source of LD Frobenius algebras in GV categories

Theorem (Fuchs, Schaumann, S, Wood)

Let $m \in \widehat{\mathcal{M}}^{\otimes}$. For every choice of isomorphism $p\colon m \to Sm$ in $\mathcal{M}$, $\underline{\mathrm{Hom}}(m,m)$ is a LD Frobenius algebra in $\mathcal{C}$ with Frobenius form

$$\lambda : \ \underline{\mathrm{Hom}}(m,m) \xrightarrow{\underline{\mathrm{Hom}}(m,p)} \underline{\mathrm{Hom}}(m,Sm) \xrightarrow{\mathrm{tr}_m} K\,.$$

where

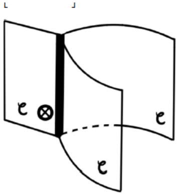
$$\mathrm{tr}_m := \left(\underline{\mathrm{Hom}}(m,S(m)) \longrightarrow G(\underline{\mathrm{Hom}}(m,m)) \xrightarrow{G(u_m)} G(1) = K \right).$$

Chapter 3

A three-dimensional graphical calculus

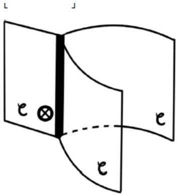
Three-dimensional graphical calculus in a strict monoidal 2-category

Tensor product as a functor $\mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ in the monoidal bicategory of finite categories:

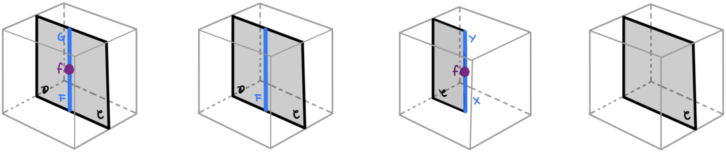


Three-dimensional graphical calculus in a strict monoidal 2-category

Tensor product as a functor $\mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ in the monoidal bicategory of finite categories:



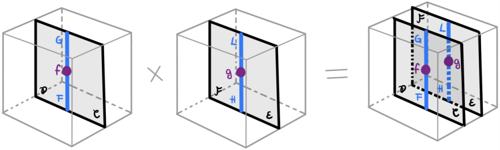
A a (labeled, stratified) surface depicts a natural transformation, here the identity. More examples:



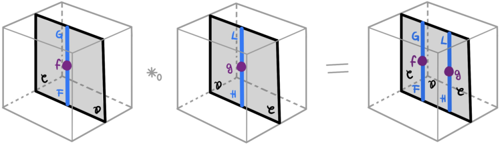
(a) $f\colon F \to G$. (b) id_F. (c) $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. (d) $\mathrm{id}_{\mathrm{id}_{\mathcal{C}}}$.

Three-dimensional graphical calculus in a strict monoidal 2-category

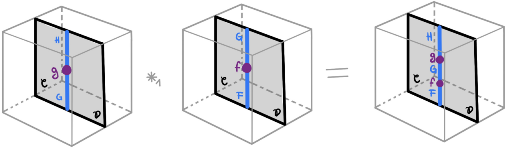
And we have three compositions:



Horizontal:

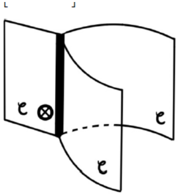


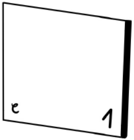
Vertical:



Three-dimensional graphical calculus

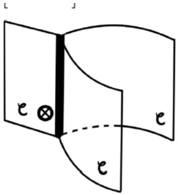
Tensor product and monoidal unit

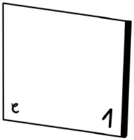




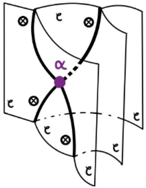
Three-dimensional graphical calculus

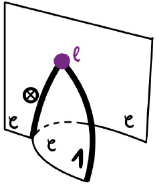
Tensor product and monoidal unit





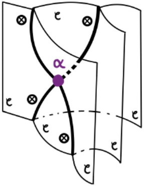
Associator and unitor

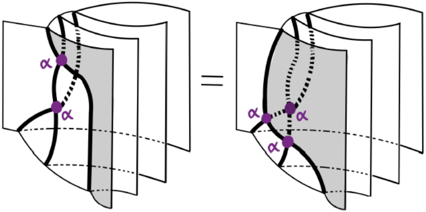




Three-dimensional graphical calculus

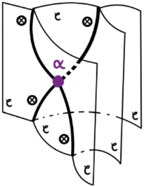
Associator and pentagon

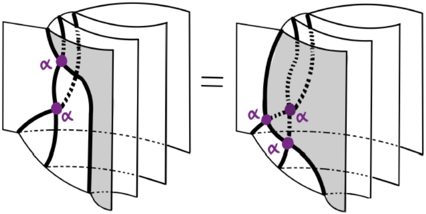




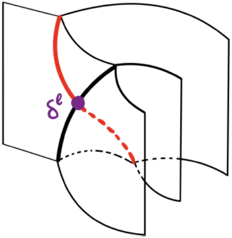
Three-dimensional graphical calculus

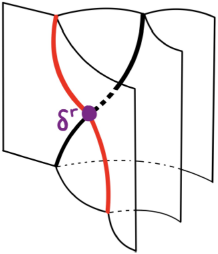
Associator and pentagon



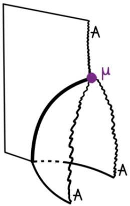


Distributors involve two tensor products, $\otimes$, black lines and $\bullet$, red lines:

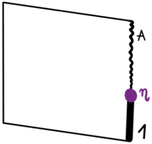




Description of algebras

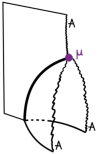


(a) Multiplication μ.

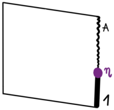


(b) Unit η.

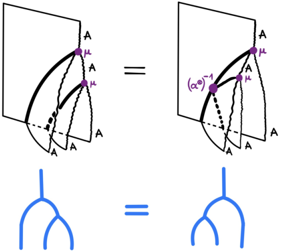
Description of algebras



(a) Multiplication μ.



(b) Unit η.



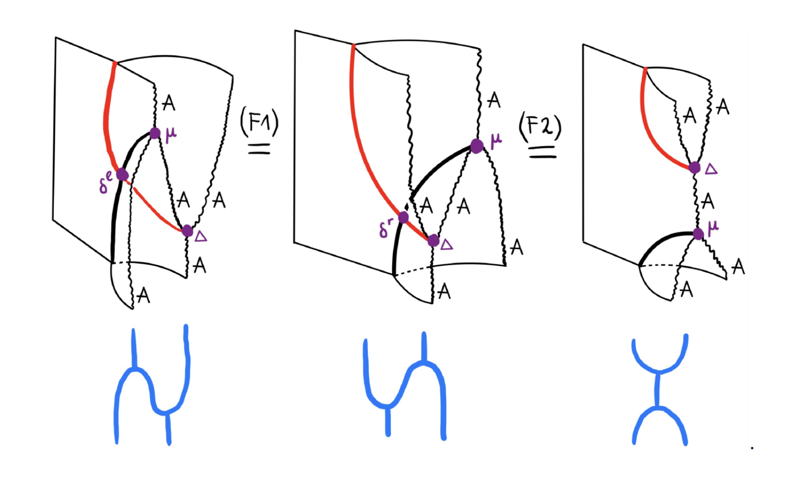
(a) Associativity.

Coassociative counital coalgebras are treated analogously.

LD Frobenius algebras

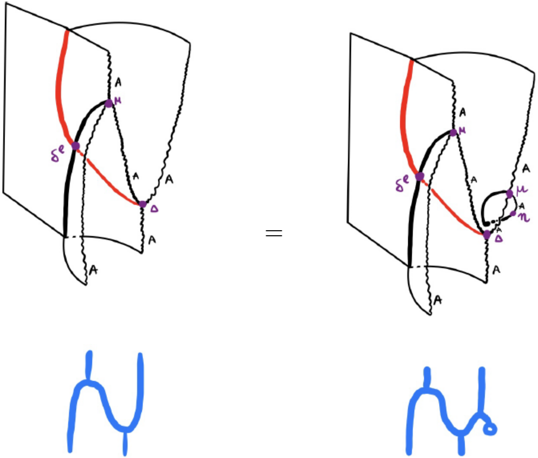
Theorem (Demirdilek, S)

The equality (F1) implies equality of the lhs and the rhs.



Proof

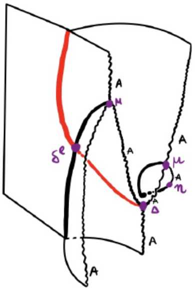
Step 1:



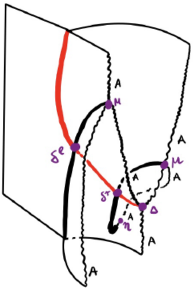
Unitality

Proof

Step 2:



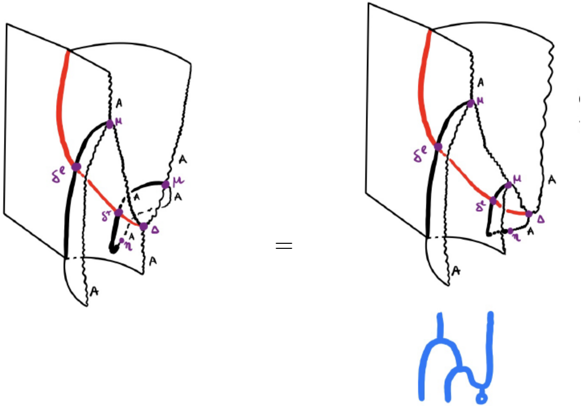
$=$



Work in the background: Triangle axiom for unitor introduces distributor δ^r

Proof

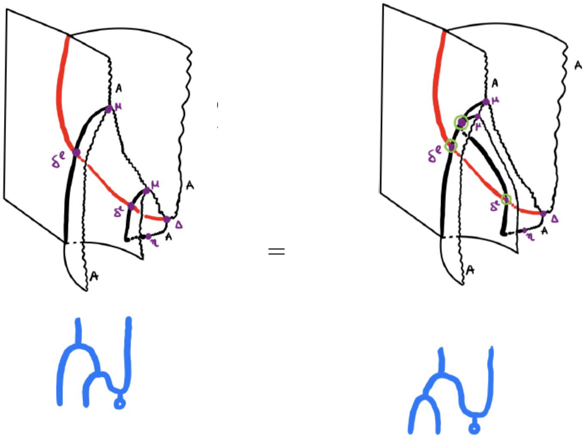
Step 3:



Frobenius axiom (F1) changes the distributor

Proof

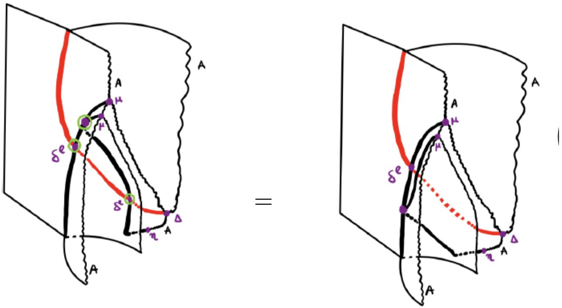
Step 4:



Associativity of multiplication introduces an associator

Proof

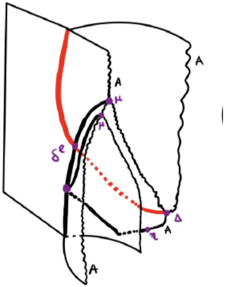
Step 5:



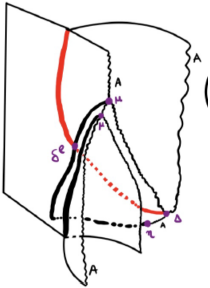
Pentagon for distributor δ^l and $\otimes$-associator

Proof

Step 6:



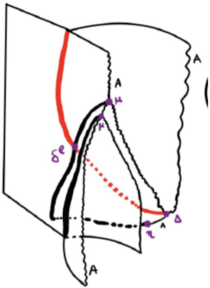
=



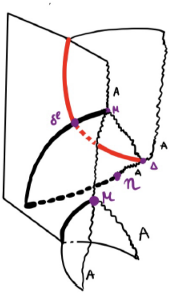
Triangle for $\otimes$

Proof

Step 7:



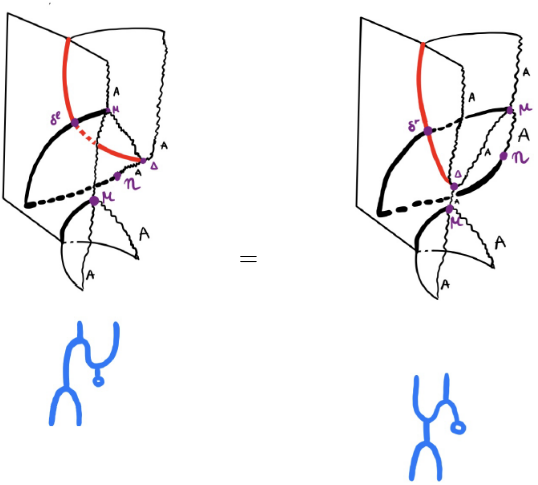
=



Strict monoidal 2-category

Proof

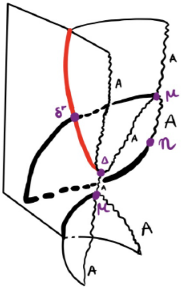
Step 8:



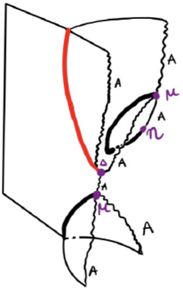
Frobenius axiom (F1) changes the distributor

Proof

Step 9:



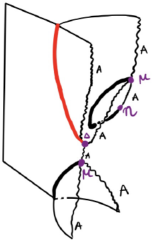
$=$



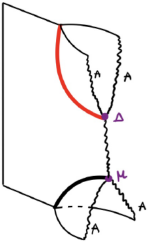
Triangle for unitor erases distributor

Proof

Step 10:



=



Unitality

Summary and outlook

Summary

- Grothendieck-Verdier duality is a natural duality structure.
- A theory of module categories, relative Serre functors, Frobenius algebras, representations of module categories as modules over algebras exists.
- Non-invertible coherence data appear and can be handled, e.g. in a three-dimensional graphical calculus

Summary and outlook

Summary

- Grothendieck-Verdier duality is a natural duality structure.
- A theory of module categories, relative Serre functors, Frobenius algebras, representations of module categories as modules over algebras exists.
- Non-invertible coherence data appear and can be handled, e.g. in a three-dimensional graphical calculus

Outlook

- Coherence theorem for associators and distributors (but not for unitors)
- Develop more methods for quantum topology, including skein theoretic methods, with a view towards CFT correlators.