Lie algebroids, groupoids and Hopf algebroids: A brief introduction.

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- The *action groupoid* is a groupoid of the form $(X \times G, X)$ where X a right G-set. The source is the action while the target is the first projection.
- Given any set X and any group G, then the pair $\big(X \times G \times X, X\big)$ is a transitive groupoid whose source and target are the third and the first projections, respectively. Here G is the isotropy type group.

The four square Loyld's Puzzle: The groupoid $\mathcal{L}_2$.

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This is a game which consist in a 2×2 chessboard with positions numbered from 1 to 4, and with 3 square pieces that can be moved at each step of the game to a nearest position, provided that is empty. Hence, each "move" represents the "state of the game", it is reversible and it can be undone in the next step.

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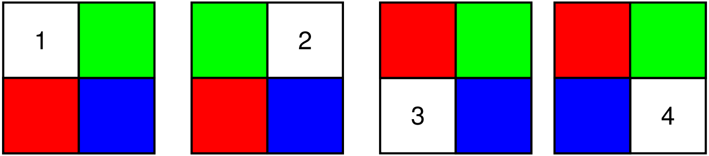

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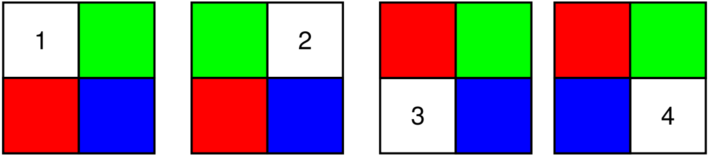

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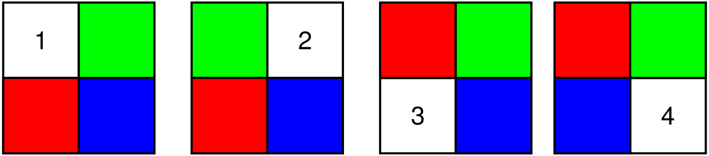

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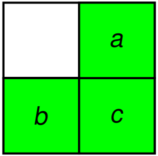

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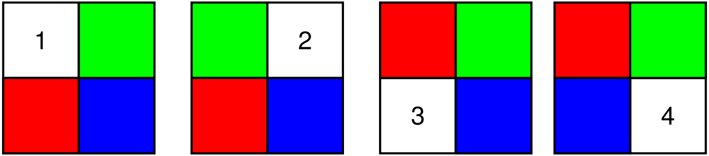

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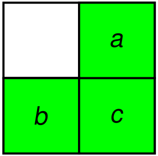

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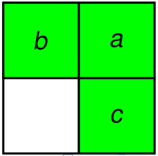

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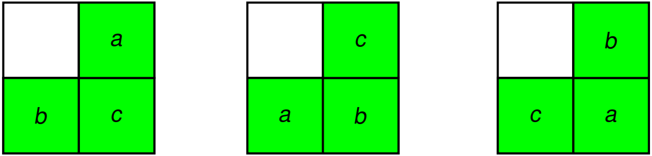

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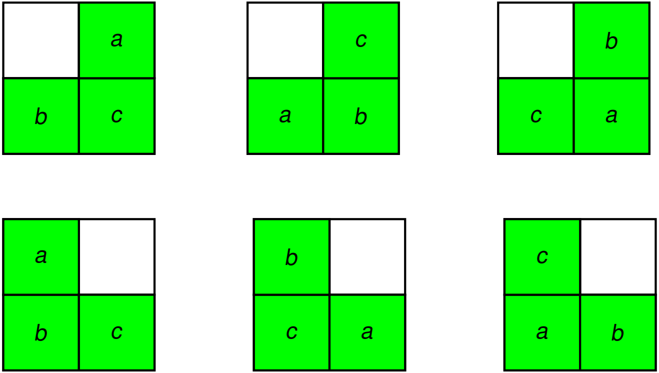

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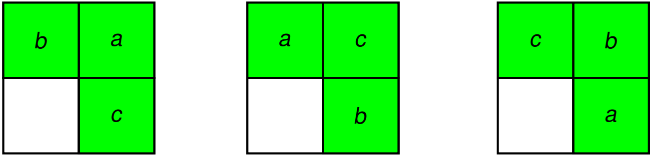

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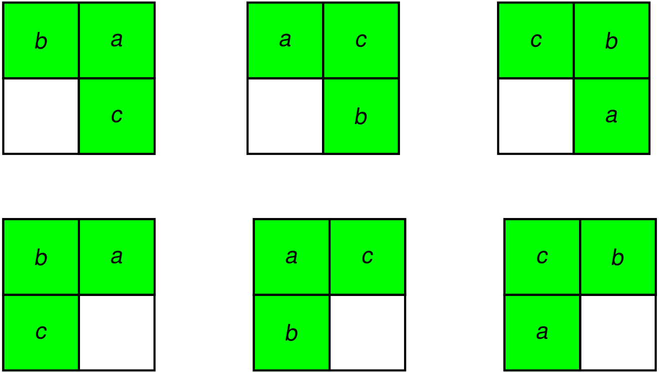

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Each move depends on its initial and final states and it is determined by a certain permutation of $\{1, 2, 3, 4\}$. Thus, we have that $\mathcal{G}_1 \subseteq \mathcal{G}_0 \times S_4 \times \mathcal{G}_0$. The resulting move out of two consecutive moves in the game is in fact the composition of the corresponding two arrows in the groupoid $\big(\mathcal{G}_0 \times S_4 \times \mathcal{G}_0, \mathcal{G}_0\big)$. The pair $(\mathcal{G}_1, \mathcal{G}_0)$ is the clearly a transitive sub-groupoid of $\big(\mathcal{G}_0 \times S_4 \times \mathcal{G}_0, \mathcal{G}_0\big)$.

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The isotropy type group of $(\mathcal{G}_0, \mathcal{G}_1)$ is the abelian group of alternating three elements $\mathcal{A}_3$. For instance,

$$\mathcal{G}^{\mathfrak{s}_1} = \Big\{\big(1, id_3, 1\big), \big(1, (234), 1\big), \big(1, (243), 1\big)\Big\},$$

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The rest of arrow from state to a state can be all computed and they are in total 48. For example, the set of arrows from $\mathfrak{s}_2$ to $\mathfrak{s}_4$ is

$$\mathcal{G}(\mathfrak{s}_2, \mathfrak{s}_4) = \Big\{\big(4, (24), 2\big), \big(4, (1342), 2\big), \big(4, (1423), 2\big)\Big\}.$$

More examples of groupoids: The Hydrogen Electron Transition.

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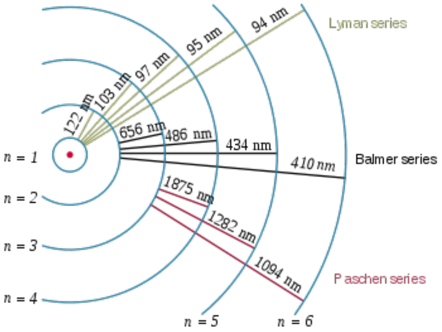

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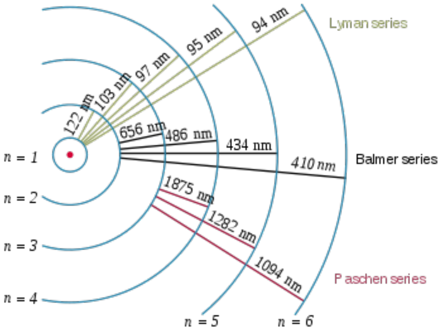

Spectral lines of the Hydrogen Atom

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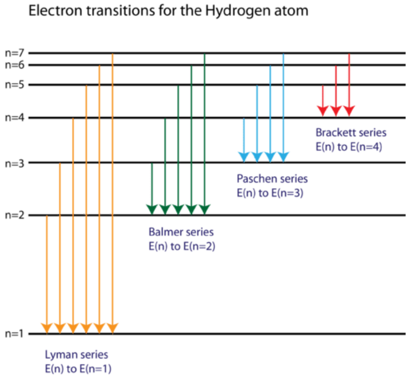

The different levels of energies $E(n)_{1\leq n\leq 7}$, form a groupoids of pairs. It seems that Alain Connes was the first who observed this, and this was perhaps one of his motivation to formulate his *non commutative geometry*.

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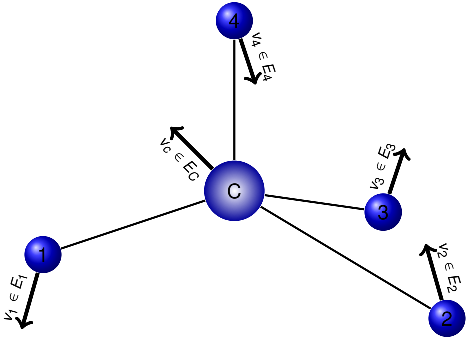

Figure: Molecular model of Carbon Tetrachloride.

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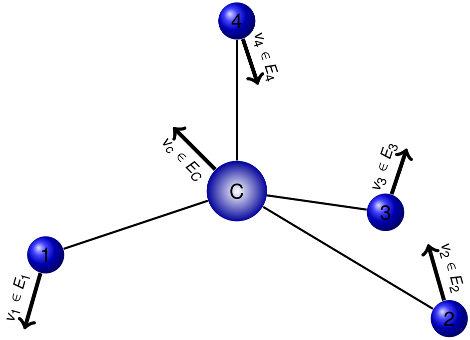

Figure: Molecular model of Carbon Tetrachloride.

In a small displacement from equilibrium, each of the atoms moves in its own three-dimensional vector space: E_1, E_2, E_3, E_4 and E_C.

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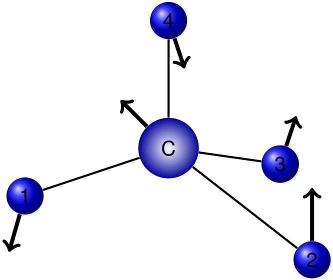

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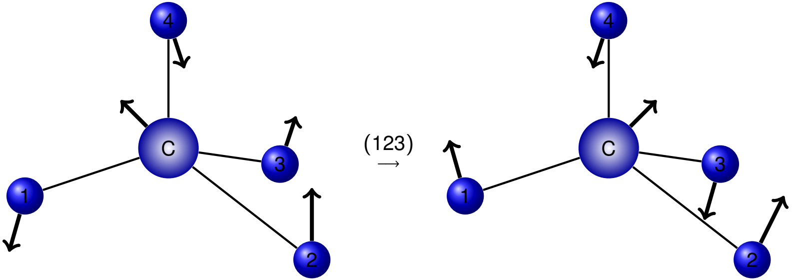

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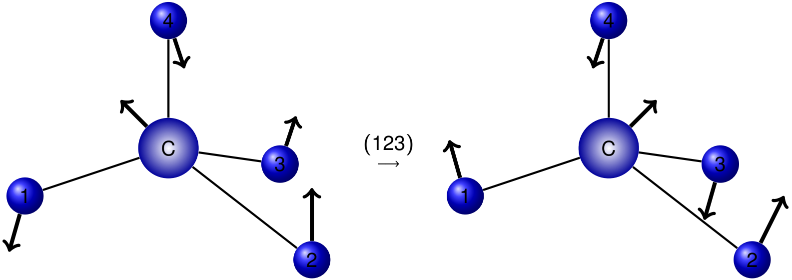

Figure: The action of the element $(123) \in S_4$ on the displacements of Carbon Tetrachloride.

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Molecular vibrations and vector bundle. Set $M = \{1, 2, 3, 4, C\}$ to be the set of atoms. Then $(\mathcal{E}, \pi)$, where $E = \biguplus_{x \in M} E_x$ and $\pi : E \to M$ is the obvious maps, is an S_4-equivariant vector bundle, or *homogeneous vector bundle*, whose associated module of global sections:

$$\Gamma(\mathcal{E}) := \left\{ \sigma : M \to E \,|\, \pi \circ \sigma = \text{identity} \right\}$$

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As we will see below, in general if we assume that a group G is acting on set M and consider it associated *action groupoid* $\mathcal{G} := (G \times M, M)$; then any G-equivariant vector bundle over M leads to a linear representation on $\mathcal{G}$. The converse also holds true, thus, any finite-dimensional (having the same dimension at each fibre) linear representation of $\mathcal{G}$, gives rise to a $\mathcal{G}$-equivariant vector bundle.

Groupoids: Finite dimensional representations.

Homogeneous vector bundles.

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we consider the category of all *$\mathcal{G}$-representations* as the symmetric monoidal $\Bbbk$-linear abelian category of functors $\left[\mathcal{G}, \mathit{Vect}_{\Bbbk}\right]$ with identity object $\mathbf{1} : \mathcal{G}_0 \to \mathit{Vect}_{\Bbbk}, x \to \Bbbk, g \to 1_{\Bbbk}$.

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The disjoint union of all the fibres of a $\mathcal{G}$-representation $\mathcal{V}$ is denoted by $\overline{\mathcal{V}} = \bigcup_{x \in G_0} \mathcal{V}_x$ and *the canonical projection* by $\pi_{\mathcal{V}}: \overline{\mathcal{V}} \to \mathcal{G}_0$. This called *the associated vector $\mathcal{G}$-bundle of the representation $\mathcal{V}$*.

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If $\mathcal{G} = (G \times M, M)$ is an action groupoid, then there is an equivalence of (symmetric monoidal) categories between the category of $\mathcal{G}$-equivariant vector bundles over M and that of linear representations of $\mathcal{G}$.

The dimensional function.

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$$d_{\mathcal{V}} : \mathcal{G}_0 \longrightarrow \mathbb{N}, \quad \left(x \longmapsto dim_{\Bbbk}\left(\mathcal{V}_x\right)\right),$$

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Let $\mathcal{V}$ and $\mathcal{W}$ be two representations in $\mathrm{rep}_{\Bbbk}(\mathcal{G})$. Then

$$d_{\mathcal{V}\oplus\mathcal{W}} = d_{\mathcal{V}} + d_{\mathcal{W}}, \quad d_{\mathcal{D}\mathcal{V}} = d_{\mathcal{V}}, \quad \text{and} \quad d_{\mathcal{V}\otimes\mathcal{W}} = d_{\mathcal{V}}\, d_{\mathcal{W}}.$$

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Therefore, the category $\mathrm{rep}_{\Bbbk}(\mathcal{G})$ is a symmetric rigid monoidal $\Bbbk$-linear abelian category. But NOT *locally finite*, in general.

Groupoids: Finite dimensional representations.

Example of representations.

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Consider the set $X = \{1, 2\}$ and denote by $\mathcal{G}^{\{1,2\}}$ the associated groupoid of pairs. Thus $\mathcal{G}_0 = \{1, 2\}$ and $\mathcal{G}_1 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

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$$rep_{\mathbb{k}}(\mathcal{G}^{\{1,2\}})\Big((n, N),\, (m, M)\Big) = M_{m,\,n}(\mathbb{k}),$$

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The other operations in $rep_{\mathbb{k}}(\mathcal{G}^{\{1,2\}})$ are

$$(n, N) \oplus (m, M) \ = \ \left(n + m, \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}\right), \quad \mathcal{D}(n, N) \ = \ (n, N^t)$$

$$(n, N)\otimes(m, M) = \Big(nm,\ (N\, b_{ij})_{1 \leq i,j \leq m}\Big), \text{ where } M = (b_{ij}), \text{ and } \mathbf{1} = (1, 1).$$

$$Tr(n, N) \ = \ n.$$

Groupoids: Finite dimensional representations.

The transitive case.

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Moreover, $rep_{\Bbbk}(\mathcal{G})$ admits a non trivial fibre functor to the category of finite dimensional vector spaces. Namely, fix an object $x \in \mathcal{G}_0$, and consider the functor

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Summarizing $(rep_{\Bbbk}(\mathcal{G}), \boldsymbol{\omega}_x)$ is a (neutral) Tannakian category in the sense of Saavedra-Rivano, Deligne and Milne.

Groupoids: Finite dimensional representations.

The fibre functor on $\operatorname{rep}_{\Bbbk}(\mathcal{G})$.

Groupoids: Finite dimensional representations.

The fibre functor on $\mathrm{rep}_{\mathbb{k}}(\mathcal{G})$. Let $\mathcal{G}$ be a groupoid and denote by $A_0(\mathcal{G}) := \mathbb{k}^{\mathcal{G}_0}$ its *base algebra* and by $A_1(\mathcal{G}) := \mathbb{k}^{\mathcal{G}_1}$ its *total algebra*. By reflecting the groupoid structure of $\mathcal{G}$, we have a diagram of algebras:

$$
\begin{tikzcd}
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Let $\mathcal{V}$ be a finite dimensional $\mathcal{G}$-representation and denote by $d_{\mathcal{V}}(\mathcal{G}_0) := \Big\{ n_1, n_2, \cdots, n_N \Big\}$ ordered as $n_1 < n_2 < \cdots < n_N$ (where obviously the maximal and minimal indices depend upon $\mathcal{V}$).

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The set of objects $\mathcal{G}_0$ is then a disjoint union $\mathcal{G}_0 = \bigcup_{i=1}^{N} G^i_{\mathcal{V}}$, where each of the $G^i_{\mathcal{V}}$'s is the inverse image $G^i_{\mathcal{V}} := d_{\mathcal{V}}^{-1}\Big(\{n_i\}\Big)$, for any $i = 1, \cdots, N$.

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This leads to a decomposition of the base algebra $A_0(\mathcal{G})$:

$$A_0(\mathcal{G}) = B_1 \times \cdots\cdots \times B_N,$$

where each of B_i's is the algebra of functions on $G^i_{\mathcal{V}}$.

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Groupoids: Finite dimensional representations.

The fibre functor on $\mathrm{rep}_{\Bbbk}(\mathcal{G})$. We can then define the functor which acts on objects by:

$$\boldsymbol{\omega} : \mathrm{rep}_{\Bbbk}(\mathcal{G}) \longrightarrow proj(A_0(\mathcal{G})), \quad \mathcal{V} \longrightarrow P_{\mathcal{V}} = B_1^{n_1} \times \cdots \times B_N^{n_N}$$

an $A_0(\mathcal{G})$-module which corresponds to the above decomposition.

Groupoids: Finite dimensional representations.

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$$\boldsymbol{\omega} : \mathrm{rep}_{\Bbbk}(\mathcal{G}) \longrightarrow proj(A_0(\mathcal{G})), \quad \mathcal{V} \longrightarrow P_{\mathcal{V}} = B_1^{n_1} \times \cdots \times B_N^{n_N}$$

an $A_0(\mathcal{G})$-module which corresponds to the above decomposition.

Now, by considering the associated vector $\mathcal{G}$-bundle of $\mathcal{V}$, we can perform the $\Bbbk$-vector space of "global sections":

$$\Gamma(\overline{\mathcal{V}}) := \Big\{ s : \mathcal{G}_0 \to \overline{\mathcal{V}} \,|\; \pi_{\mathcal{V}} \circ s = id_{\mathcal{G}_0} \Big\}.$$

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The functor $\boldsymbol{\omega}$ is a non trivial exact, faithful and symmetric monoidal functor. It is termed *the fibre functor* of $\text{rep}_{\Bbbk}(\mathcal{G})$.

Lie algebroids: Definition and examples

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Let $\mathcal{M}$ be a connected smooth real (or almost complex) manifold and $A := \mathcal{C}^\infty(\mathcal{M})$. Consider $(\mathcal{L}, \mathcal{M})$ a locally trivial vector bundle with a constant rank. Denote by $L := \Gamma(\mathcal{L})$ its A-module of smooth global sections. In this case, this is a finitely generated and projective module with a constant rank.

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The pair $(\mathcal{L}, \mathcal{M})$ is called a *Lie algebroid*, provided that there exist a morphism of vector bundles $\varphi : \mathcal{L} \to T\mathcal{M}$ and a structure of Lie algebra on L, such that $\Gamma(\varphi) : L \to \Gamma(T\mathcal{M})$ is a Lie algebras morphisms satisfying:

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In a more general fashion, a *Lie-Rinehart algebra*, is a pair (L, A) consisting of an algebra A and an A-module L with a Lie algebra (over $\mathbb{k}$) structure together with a Lie algebras map $\phi : L \to Der_{\mathbb{k}}(A)$ (*the anchor*) which is A-linear and satisfies:

$$[X, aY] \;=\; a[X, Y] + \phi(X)(a)Y$$

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- (The Lie algebroid of a Lie groupoid) Let us consider a Lie groupoid

$$\mathcal{G}: \quad \begin{tikzcd} \mathcal{G}_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathcal{G}_0 \arrow[l, "\iota" description] \end{tikzcd},$$

where $\mathcal{G}_1$ is assumed to be a connected smooth real manifold and s, t are surjective submersions. Consider the following vector bundle $\mathcal{E} = \cup_{x \in \mathcal{G}_0} \mathcal{E}_x$, where each fibre $\mathcal{E}_x$ is the $\mathbb{R}$-vector space $\mathcal{E}_x = Der^{s^*}_{\mathbb{R}}(\mathcal{C}^{\infty}(\mathcal{G}_1), \mathbb{R}_{\iota(x)}) \cong Der_{\mathbb{R}}(\mathcal{C}^{\infty}(\mathcal{G}_x), \mathbb{R}_{\iota(x)})$. Then $(\Gamma(\mathcal{E}), \mathcal{C}^{\infty}(\mathcal{G}_0))$ has a structure of Lie-Rinehart algebra.

Lie algebroids: representations and fibre functor

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Let (L, A) be a Lie-Rinehart algebra. An *L-representation* is a pair (M, ρ), where M is an A-module and $\rho : L \to End_{\Bbbk}(M)$ is simultaneously a morphism of A-modules and Lie algebras such that

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We denote by $rep_{\Bbbk}(L)$ the full subcategory of L-representations with finitely generated and projective underlying A-modules. The forgetful functor, leads then to a functor $\omega : rep_{\Bbbk}(L) \to proj(A)$.

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The category $rep_{\Bbbk}(L)$ is a $\Bbbk$-linear symmetric and rigid monoidal category with identity object $\mathbb{I} = (A, \phi)$, whose endomorphism ring coincides with the sub-algebra $A^c \subset A$ of L-constants elements:

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In the particular case $(L = \mathbb{C}.\partial_z, \mathbb{C}[z])$, we have that $rep_{\mathbb{C}}(L)$ coincides with the category of differential $\mathbb{C}[z]$-modules (i.e., *linear differential matrix equations*).

Hopf Algebroids: Definition and examples

Commutative Hopf algebroids:

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Thus, we are considering a pair of objects $(A, \mathcal{H})$ in $\mathscr{A}\!ff_{\Bbbk}$ such that, for any other object C, we have, in a funtorial, way a structure of groupoid (over the fibres)

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$$A \overset{s}{\underset{t}{\underset{\longleftarrow\varepsilon}{\rightrightarrows}}} \mathcal{H}, \qquad \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes_A \mathcal{H}, \qquad {}_s\mathcal{H}_t \xrightarrow{\mathcal{S}} {}_t\mathcal{H}_s.$$

source, target and the identity arrow — composition — inverse arrow

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Morphism of Hopf algebroids: A pair of algebra maps $(\phi_0, \phi_1) : (A, \mathcal{H}) \to (B, \mathcal{K})$ is said to be a *morphism of Hopf algebroids*, if ϕ_0 and ϕ_1 are compatible with both Hopf structures, that is, they induce a morphism $\Phi : \mathbb{K} \to \mathbb{H}$ between the associated presheaves of groupoids.

Hopf Algebroids: Definition and examples.

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The comultiplication $\Delta : {}_s\mathcal{H}_t \longrightarrow {}_s\mathcal{H}_t \otimes_A {}_s\mathcal{H}_t$ is given by:

$$\Delta(x) = x \otimes_A 1, \quad \Delta(y) = 1 \otimes_A y, \text{ and for } n \geq 1 :$$

$$\Delta(y_n) = \sum_{\substack{(k_1, k_2, \cdots, k_n) \\ k_1 + 2k_2 + \cdots + nk_n = n}} \frac{n!}{k_1! \cdots k_n!} \left(\left(\frac{y_1}{1!}\right)^{k_1} \left(\frac{y_2}{2!}\right)^{k_2} \cdots \left(\frac{y_n}{n!}\right)^{k_n} \right) \otimes_A y_{k_1 + k_2 + \cdots + k_n},$$

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Hopf algebroids: Their categories of comodules

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Any morphism $\phi : (A, \mathcal{H}) \to (B, \mathcal{K})$ of Hopf algebroids induces a symmetric monoidal functor (the *induction functor*):

$$\phi^* : Comod_{\mathcal{H}} \longrightarrow Comod_{\mathcal{K}}$$

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$(A, \mathcal{H})$ is said to be a geometric Hopf algebroid, provided that $\mathcal{H}$ is a flat A-module and can be reconstructed from its category of geometric comodules via the forgetful functor $\mathcal{O}$. In other words, $(A, \mathcal{H})$ *is* $comod_{\mathcal{H}}^{G}$*-Galois*.

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Next we will give another class of examples of geometric Hopf algebroids.

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It turns out that this functor lands in the full subcategory of geometric A-modules. Thus, we have a commutative diagram

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Representative functor.

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So far, we have construct a contravariant functor $\mathscr{R}_{\Bbbk} : \mathit{Grpd} \to \mathit{GHAlgd}$, with a commutative diagram:

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FIRST RESULT:

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First result: Both functors establish contravariant adjuntions:

$$\begin{tikzcd} \mathit{Grpd} \arrow[r, shift left, "\mathscr{R}_{\Bbbk}"] \arrow[r, phantom, "\perp"] & \mathit{GHAlgd} \arrow[l, shift left, "\chi_{\Bbbk}"] \\ \mathit{TGrpd} \arrow[u, hook] \arrow[r, shift left, "\mathscr{R}_{\Bbbk}"] \arrow[r, phantom, "\perp"] & \mathit{GTHAlgd} \arrow[u, hook] \arrow[l, shift left, "\chi_{\Bbbk}"] \end{tikzcd}$$

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We consider the following two vector spaces:

$$\mathrm{Der}_{\mathcal{H}}{}^{s}(\mathcal{H}, \mathcal{H}) := \left\{ \begin{matrix} \delta \in \mathrm{Hom}_{\Bbbk}(\mathcal{H}, \mathcal{H}) \,|\, \delta \circ s = 0,\ \delta(uv) = \delta(u)v + u\delta(v), \\ \Delta(\delta(u)) = u_1 \otimes_A \delta(u_2), \ \text{ for all } u, v \in \mathcal{H} \end{matrix} \right\},$$

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$$\mathrm{Der}_{\Bbbk}{}^{s}(\mathcal{H}, A_{\varepsilon}) := \left\{ \begin{matrix} \delta \in \mathrm{Hom}_{\Bbbk}(\mathcal{H}, A) \mid \delta \circ s = 0, \\ \delta(uv) = \delta(u)\varepsilon(v) + \varepsilon(u)\delta(v), \quad \text{for all } u, v \in \mathcal{H} \end{matrix} \right\}.$$

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The Lie-Rinehart algebra of a Hopf algebroid: Let $(A, \mathcal{H})$ be a Hopf algebroid over a ground field $\Bbbk$ and denote by $\mathcal{I} := \ker(\varepsilon)$ its augmentation ideal. We consider A as an $\mathcal{H}$-module via the counit algebra map and denote this module by A_ε.

We consider the following two vector spaces:

$$\mathrm{Der}_{\mathcal{H}}{}^{s}(\mathcal{H},\, \mathcal{H}) := \left\{ \begin{array}{c} \delta \in \mathrm{Hom}_{\Bbbk}(\mathcal{H},\mathcal{H}) \mid \delta \circ s = 0,\ \delta(uv) = \delta(u)v + u\delta(v), \\ \Delta(\delta(u)) = u_1 \otimes_A \delta(u_2),\quad \text{for all } u, v \in \mathcal{H} \end{array} \right\},$$

and

$$\mathrm{Der}_{\Bbbk}{}^{s}(\mathcal{H},\, A_\varepsilon) := \left\{ \begin{array}{c} \delta \in \mathrm{Hom}_{\Bbbk}(\mathcal{H}, A) \mid \delta \circ s = 0, \\ \delta(uv) = \delta(u)\varepsilon(v) + \varepsilon(u)\delta(v),\ \text{for all } u, v \in \mathcal{H} \end{array} \right\}.$$

We have a commutative diagram of A-modules:

$$\begin{tikzcd} {}^{*}\mathcal{H} \arrow[r, "\cong"] & \mathrm{End}^{\mathcal{H}}(\mathcal{H}) \\ \mathrm{Der}_{\Bbbk}{}^{s}(\mathcal{H},\, A_\varepsilon) \arrow[u, hook] \arrow[r, dashed, "\cong"] & \mathrm{Der}_{\mathcal{H}}{}^{s}(\mathcal{H},\, \mathcal{H}). \arrow[u, hook] \end{tikzcd}$$

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The Lie-Rinehart algebra of a Hopf algebroid: Moreover, the A-module $\mathrm{Der}_{\Bbbk}{}^{s}(\mathcal{H}, A_{\varepsilon})$ admits a structure of Lie $\Bbbk$-algebra with bracket

$$[\delta, \delta'] := \delta * \delta' - \delta' * \delta : \mathcal{H} \longrightarrow A_{\varepsilon}, \Big(u \longmapsto \delta\big(u_1 t(\delta'(u_2))\big) - \delta'\big(u_1 t(\delta(u_2))\big)\Big)$$

and this structure can be transferred to ${}^{*}\left(\frac{\mathcal{I}}{\mathcal{I}^2}\right)$ in a unique way.

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The pair $(A, \mathrm{Der}_{\Bbbk}{}^{s}(\mathcal{H}, A_{\varepsilon}))$ admits a structure of Lie-Rinehart algebra with anchor map:

$$\omega : \mathrm{Der}_{\Bbbk}{}^{s}(\mathcal{H}, A_{\varepsilon}) \longrightarrow \mathrm{Der}_{\Bbbk}(A), \quad \Big(\delta \longmapsto \delta \circ t\Big).$$

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Fix an algebra A and denote by $HAlgd_A$ the category of all Hopf algebroids with base algebra A, and by $LieRin_A$ the category of all Lie-Rinehart algebras with base algebra A. We have then construct a contravariant functor:

$$HAlgd_A \xrightarrow{\quad \mathscr{L} \quad} LieRin_A$$

referred to as *the differentiation functor*.

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Example of Lie-Rinehart algebra of a given Hopf algebroid:

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(•) Consider the Malgrange's Hopf algebroid $(A, \mathcal{H})$ over $\mathbb{C}$ and with $A = \mathbb{C}[X]$.

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$$\omega : A^{\mathbb{N}} \longrightarrow Der_{\mathbb{C}}(A), \quad \Big(\mathfrak{a} := (a_n)_{n \in \mathbb{N}} \longmapsto \big(p \mapsto a_0 \partial p\big)\Big)$$

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$$\big[\mathfrak{a}, \mathfrak{b}\big]_0 = a_0\partial b_0 - b_0 \partial a_0, \quad \big[\mathfrak{a}, \mathfrak{b}\big]_1 = a_0 \partial b_1 - b_0 \partial a_1,$$

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($\bullet$) Assume now, we are given an affine $\mathbb{k}$-group $\mathcal{G} := Alg_{\mathbb{k}}(H, -)$ acting on an affine $\mathbb{k}$-scheme $\mathcal{X} := Alg_{\mathbb{k}}(A, -)$. There is a well known anti-homomorphism of Lie algebras $L := \mathcal{L}ie(\mathcal{G})(\mathbb{k}) \to \mathrm{Der}_{\mathbb{k}}(\mathscr{O}_{\mathbb{k}}(\mathcal{X}))$.

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Formal integration of Lie-Rinehart algebras.

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The first integration functor:

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The first integration functor: Fix a Lie-Rinehart algebra (A, L) and denote by $\mathcal{U}_A(L)$ its universal enveloping (right) Hopf algebroid. This is a co-commutative Hopf algebroid whose category of right $\mathcal{U}_A(L)$-modules with finitely generated and projective underlying A-modules coincides with the rigid and symmetric monoidal category $rep_A(L)$ of L-representations.

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SECOND RESULT:

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Second Result: Assume that A satisfies the property (Pzeta), then there is a contravariant adjunction

$$\begin{tikzcd} LieRin_A \arrow[r, shift left, "\mathscr{I}"] \arrow[r, phantom, "\perp"] & GalHAlgd_A \arrow[l, shift left, "\mathscr{L}"] \end{tikzcd}$$

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The second integration functor: Fix A as before to be a base algebra. By applying the Special Adjoint Functor Theorem (SAFT) to the category of A-rings, one can construct a contravariant functor:

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together with a natural transformation $\tilde{\zeta}_L : \mathcal{U}_A(L)^\circ \to \mathcal{U}_A(L)^\bullet := \mathscr{I}'(L)$ that fits in the following commutative diagram:

$$\begin{tikzcd} \mathcal{U}_A(L)^\circ \arrow[rr, "\zeta"] \arrow[dr, "\tilde{\zeta}"'] & & \mathcal{U}_A(L)^* \\ & \mathcal{U}_A(L)^\bullet \arrow[ur, "\xi"'] & \end{tikzcd}$$

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The map $\tilde{\zeta}$ is an equality, when $A = \Bbbk$, and the whole diagram reduces to equalities when $\mathcal{U}_A(L)_A$ is finitely generated and projective module.

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Moreover, for any Lie-Rinehart algebra (A, L) we have a commutative diagram of natural transformations:

$$\begin{tikzcd} L \arrow[r, "\Theta_L"] \arrow[dr, "\Theta'_L"'] & \mathscr{L}(\mathcal{U}_A(L)^\circ) \\ & \mathscr{L}(\mathcal{U}_A(L)^\bullet). \arrow[u, "\mathscr{L}(\hat{\zeta})"'] \end{tikzcd}$$

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Integrating Lie-Rinehart algebras: Now we can address the *integration problem* for Lie-Rinehart algebra in general and hence for Lie algebroids in particular.

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Integrating Lie-Rinehart algebras: Now we can address the *integration problem* for Lie-Rinehart algebra in general and hence for Lie algebroids in particular.

Given a Lie-Rinehart algebra (A, L) such that L_A is finitely generated and projective module with constant rank. Under which conditions (on both A and L), one can construct a Hopf algebroid $(A, \mathcal{H})$ such that $L \cong \mathscr{L}(\mathcal{H})$?

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Thank you!